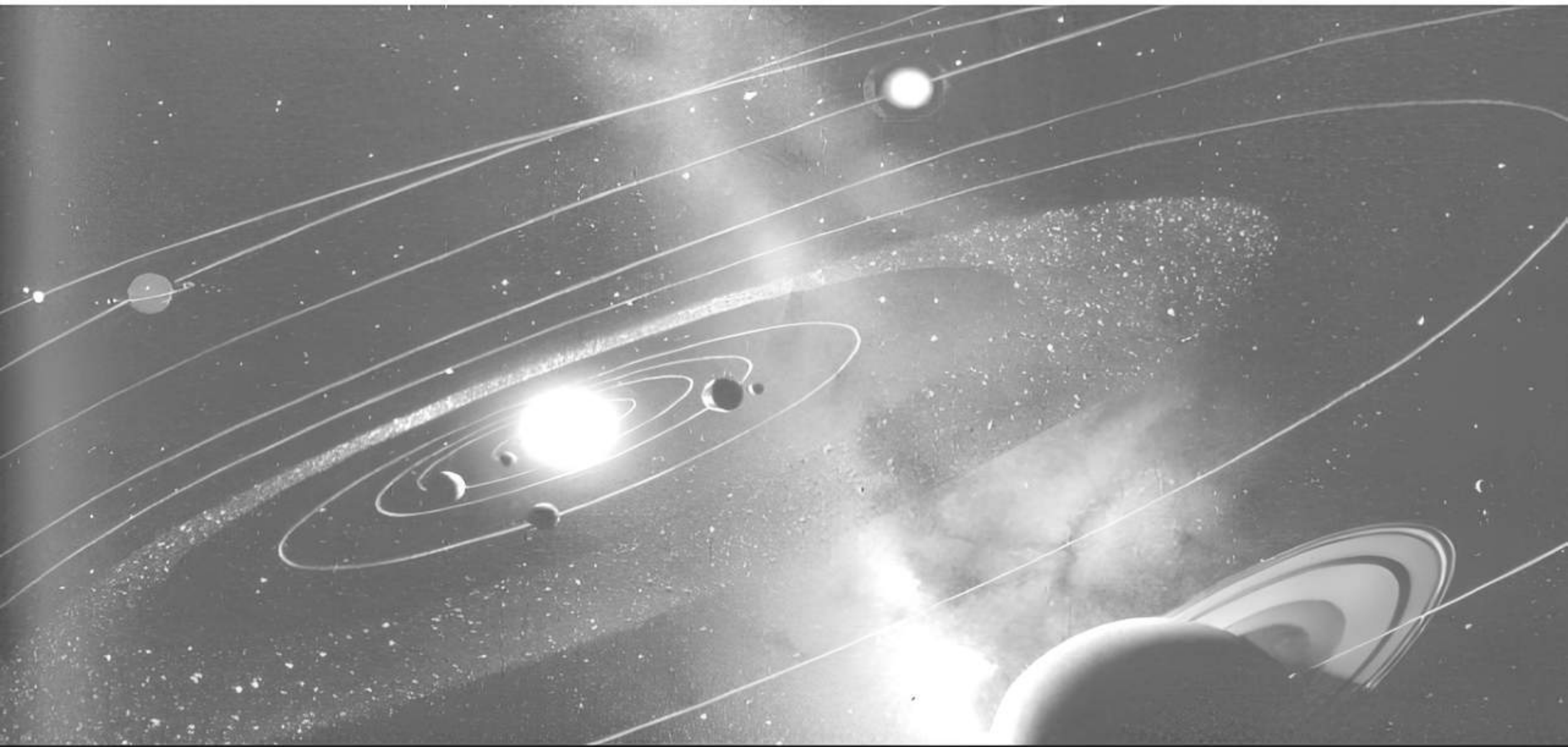
6 Systems of Particles and Rotational Motion



Since the invention of the wheel, mankind has achieved wonders. Today, an electric motor is used in almost every piece of machinery you can ever find, all working on the principle of the wheel. Even planets in galaxies move in circle. They revolve around the Sun, which is the heaviest body in the solar system. All the planets have an axis of rotation and revolution. For such fundamentals as the center of mass and axis of rotation, this chapter is an important branch of physics.

Topic Notes

- *Center of Mass and Angular Momentum*

CENTRE OF MASS AND ANGULAR MOMENTUM 1

TOPIC 1

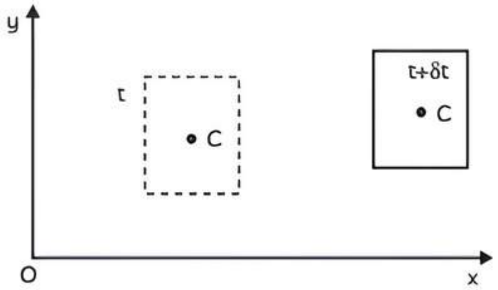
RIGID BODY

A rigid body is an assemblage of a large number of material particles that don't change their mutual distances under any circumstances or in other words, the body is not deformed under any circumstances. Actual material bodies are never perfectly rigid and deform under the action of external forces. When these deformations are small enough not to be considered during the course of motion, the body is assumed to be a rigid body. Hence, all solid objects such as stones, balls, vehicles, etc. are considered rigid bodies while analyzing their translational as well as rotational motion.

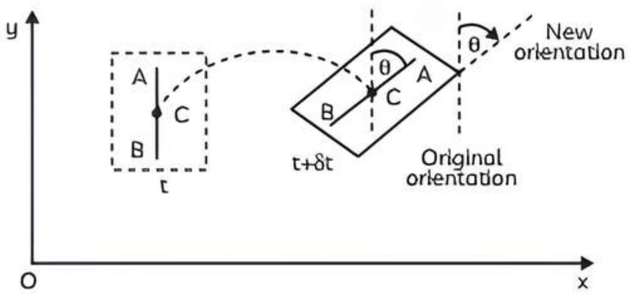
Rotational Motion of a Rigid Body

Any kind of motion is identified by change in position or change in orientation, or change in both. If a body changes its orientation during its motion, it is said to be in rotational motion.

In the following figures, a rectangular plate is shown moving in the x-y plane. The point C is its center of mass. In the first case, it does not change its orientation, therefore it is in pure translation motion. In the second case, it changes its orientation during its motion. It is a combination of translational and rotational motion.



Pure Translational Motion



Combination of Translational and Rotational motion

Rotation, *i.e.*, change in orientation, is identified by the angle through which a linear dimension or a straight line drawn on the body turns. In the figure, this angle is shown by θ.

Motion of body involving rotation can be classified into the following three categories:

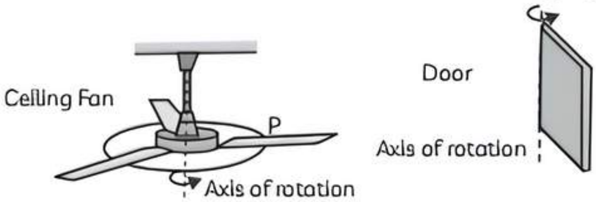
(1) Rotation about a fixed axis.

(2) Rotation about an axis in translation.

(3) Rotation about an axis in rotation.

Rotation About a Fixed Axis

In pure rotational motion, every particle of the rigid body moves in circles of different radii about a fixed line, which is known as axis of rotation. Rotation of ceiling fan, opening and closing of doors and rotation of needles of a wall clock, etc., come in this category.



When a ceiling fan rotates, the vertical rod, supposing it remains stationary and all the particles on the fan move on circular paths. The circular path of a particle *P* along its blade is shown by a dotted line. Centers of the circular path followed by every particle on the central line through the rod. The central line is known as the axis of rotation and is shown by a dashed line. All the particles on the axis of rotation are at rest, therefore, the axis is stationary, and the fan is in rotation about this fixed axis.

A door rotates about a vertical line that passes through its hinges. This vertical line is the axis of the figure, the axis of rotation is shown by the dashed line.

Axis of Rotation

An imaginary line perpendicular to the plane of circular paths of particles of a rigid body in rotation and containing the centers of all these circular paths is known as the axis of rotation.

It is not necessary that the axis of rotation pass through the body. Consider a system in which a block is fixed to a rotating disc. The axis of rotation passes through the center of the disc but not through the block.

Rotation About an Axis in Translation

Rotation about an axis in translation include a broad category of motion. A rolling motion is an example of this kind of motion.

Rotation About an Axis in Rotation

In this kind of motion, the body rotates about an axis, which in turn rotates about some other axis. As an example, consider a rotating top, the top rotates about its central axis of symmetry and this axis sweeps a cone about a vertical axis. Another example of rotation about an axis is a swinging table-fan while running. The table-fan rotates about its shaft, along which its axis of rotation passes. When running swings, its shaft rotates about a certain axis.

Example 1.1: A system of particles is called a rigid body when:

(a) any two particles of a system may have displacements in opposite directions under the action of a force

(b) any two particles of a system may have velocities on opposite directions under action of a force

(c) any two particles of a system may have a zero relative velocity

(d) any two particles of a system may have displacements in the same direction under the action of a force

Ans. *(c) any two particles of a system may have a zero relative velocity*

Explanation: A rigid body does not deform under the action of an applied force, and there is no relative motion of any two particles constituting that rigid body. So, it means that a system of particles is called a rigid body, when any two particles in the system have a zero relative velocity

TOPIC 2

CENTER OF MASS

The center of mass of a system is the point that behaves as the whole mass of the system is concentrated at it and all external forces are acting on it. For rigid bodies, the center of mass is independent of the state of the body, *i.e.*, whether it is in rest or accelerated motion, the center of mass will remain the same.

Characteristics of Center of Mass

(1) It need not hold mass physically, e.g., for a hollow sphere, the center of mass is at the geometrical center of the sphere, although there is no mass present physically at the center.

(2) Location of center of mass depends on the distribution of masses and their individual location. For regular geometrical-shaped bodies, having a uniform distribution of mass, the center of mass is located at their centers.

(3) When no external force acts on a body, the center of mass has constant velocity and constant angular momentum. Acceleration is zero.

Position of Center of Mass (C.M.) of a System of Particles

(1) Position vector of C.M. of a two particles system is given by,

$$\vec{x}_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Centre of mass of two particles of equal masses lies at the mid-point of the line joining the two masses.

If the masses are unequal, C.M. lies towards the heavier mass. If C.M. is the origin then,

$$m_1x_1 + m_2x_2 = 0$$

(2) For a n particles system, the position vector of the C.M. is given by,

If a system consists of n particles of masses m_1, m_2, m_3 ... m_n having position vectors x_1, x_2, x_3, ... x_n, then position vector of center of mass of the system.

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \dots m_nx_n}{m_1 + m_2 + m_3 + \dots m_n}$$

$$= \frac{\sum_{i=1}^{n} m_ix_i}{\sum m_i}$$

If C.M. is the origin, then

$$\sum_{i=1}^{n} m_ix_i = 0$$

If the system has a continuous distribution of mass, treating the mass element dm at position $\vec{x}_1$ as a point mass and replacing summation by integration,

so that, $$x_{CM} = \frac{1}{M}\int x\,dm,$$

$$y_{CM} = \frac{1}{M}\int y\,dm$$

and $$z_{CM} = \frac{1}{M}\int z\,dm$$

Centre of Mass of Rigid Continuous Bodies

For a real body which is a continuous distribution of matter, point masses are then differential mass elements dm and centre of mass is defined as,

$$x_{CM} = \frac{1}{M}\int x\,dm$$

$$y_{CM} = \frac{1}{M}\int y\,dm$$

and $$z_{CM} = \frac{1}{M}\int z\,dm$$

Where M is the total mass of that real body.

If we choose the origin of coordinates axes at centre of mass then,

$$\int x\,dm = \int y\,dm = \int z\,dm = 0$$

Motion of Centre of Mass

The centre of masses of a system of particles moves as if all the masses of the system were concentrated at the centre of mass and all the external forces were applied at that point. As for a system of particles, position of centre of mass is

$$Mx_{CM} = m_1 x_1 + m_2 x_2 + m_3 x_3 + ... m_n x_n \quad ...(i)$$

Velocity of CM

Differentiating equation (i) with respect to time.

Velocity about the centre of mass,

$$v_{CM} = \frac{\sum_{i=1}^{n} m_i v_i}{M}$$

Similarly, acceleration of CM, $a_{CM} = \frac{d}{dt} v_{CM}$

Acceleration about the centre of mass,

$$a_{CM} = \frac{\sum_{i=1}^{n} m_i a_i}{M}$$

Example 1.2: Give the location of the centre of mass of a:

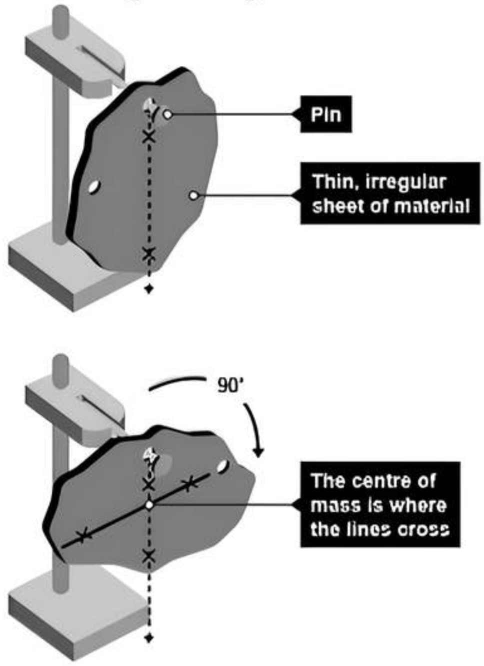
(A) Sphere (B) Cylinder
(C) Ring (D) Cube

Each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body? [NCERT]

Ans. In all four cases, as the mass density is uniform, the centre of mass is located at their respective geometrical centres. No, it is not necessary that the centre of mass of a body should lie on the body. For example, in case of a circular ring, centre of mass is at the centre of the ring, where there is no mass.

Example 1.3: Case Based:

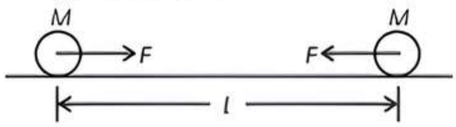
A body or a system of bodies centre of mass is the point that moves as if all of the mass were concentrated there and all external forces were applied to it. As a result, the centre of mass is a point where the total mass of a body or system of bodies is meant to be concentrated. If a system has more than one particle (or body) and the net external force on the system in one direction is zero, the system is said to be at rest. The centre of mass will then stop moving in that direction. Despite the fact that some particles in the system may flow in that direction.



(A) The centre of mass of a system of two particles divides, the distance between them:

(a) in an inverse ratio of the square of masses of particles
(b) in the direct ratio of the square of masses of particles
(c) in an inverse ratio of masses of particles
(d) in the direct ratio of masses of particles

(B) Two balls of same masses start moving towards each other due to gravitational attraction, if the initial distance between them is l. Then, they meet at:



(a) $\frac{l}{2}$ (b) l
(c) $\frac{l}{3}$ (d) $\frac{l}{4}$

(C) Two bodies of masses 1 kg and 2 kg are lying in xy-plane at (-1, 2) and (2, 4), respectively. What are the coordinates of the centre of mass?

(D) Assertion (A): The earth is slowing down and as a result the moon is coming nearer to it.

Reason (R): The angular momentum of the earth moon system is conserved.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true and R is not correct explanation of A.

(c) A is true but R is false.

(d) A is false and R is also false.

(E) Two particles A and B initially at rest move towards each other under a mutual force of attraction. At the instant, when the speed of A is v and the speed of B is 2v, then what is the speed of centre of mass of the system?

Ans. (A) *(c) in an inverse ratio of masses of particles*

Explanation: Centre of mass of a system of two particles is,

Then, $$r_{CM} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

If, $m_1 + m_2 = M$ = total mass of the particles.

$$r_{CM} = \frac{m_1 r_1 + m_2 r_2}{M}$$

Then, $$r_{CM} = \frac{1}{m}$$

So, the above relation clearly shows that the centre of mass of a system of two particles divides the distance between them in the inverse ratio of masses of particles.

(B) *(a)* $\frac{l}{2}$

Explanation: As the balls were initially at rest and the forces of attraction are internal, then their centre of mass (CM) will always remain at rest.

So, $v_{CM} = 0.$

As CM is at rest, they will meet at CM.

Hence, they will meet at $\frac{l}{2}$ from any initial positions.

(C) Let the coordinates of the centre of mass be (x, y).

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{1 \times (-1) + 2 \times 2}{3} = 1$$

$$y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

$$= \frac{1 \times 2 + 2 \times 4}{3} = \frac{10}{3}$$

Therefore, the coordinates of centre of mass are $\left(1, \frac{10}{3}\right)$.

(D) *(c) A is true but R is false.*

Explanation: The angular momentum of earth-moon system will be conserved because no torque is acting on it.

$$\text{So, } T = \frac{dL}{dT}$$

If $$T = 0, \frac{dL}{dT} = 0$$

$\Rightarrow$ L is constant.

Thus, angular momentum is constant. So reason is wrong.

So, we get $I_1\omega_1 = I_2\omega_2$

Where I_1 & I_2 are moment of inertia of earth and moon and ω_1 and ω_2 are their angular velocities. If earth slows down ω_1 will be decreased. So, I_2 will be decreased if we take ω_2 to remain constant.

Now, $I_2 = m_2 r_2^2$

Where m_2 is mass of moon and r_2 is radius of moon's orbit. So, r_2 will be reduced to reduce I_2. Hence, moon will come near to the earth. Hence, assertion is right.

(E) As per the question, two particles A and B are initially at rest, and move towards each other under a mutual force of attraction. It means that no external force is applied to the system. Therefore, $F_{ext} = 0$.

So, there is no acceleration of CM. This means the velocity of the CM remains constant. As, the initial velocity of CM, $v_i = 0$ and final velocity of CM, $v_f = 0$. So, the speed of centre of mass of the system will be zero.

TOPIC 3

LINEAR MOMENTUM AND ANGULAR VELOCITY OF SYSTEM OF PARTICLES

Linear Momentum

The linear momentum p of an object with mass m and velocity v is defined as:

$$\vec{p} = m\vec{v}$$

From this definition, it is clear that the unit of momentum is (kg. m/s) or (N.s). Since, the momentum which is related to the linear motion of the object is called linear momentum. Angular momentum, which is related to the angular motion of the object. Under certain circumstances, the linear momentum of a system is conserved. The linear momentum of a particle is related to the net force acting on that object.

$$\Sigma\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}$$

The rate of change of linear momentum of a particle is equal to the net force acting on the object and is pointed in the direction of the force. If the net force acting on an object is zero, its linear momentum is constant (conservation of linear momentum).

The total linear momentum p of a system of particles is defined as the vector sum of the individual linear momentum.

$$\vec{p} = \sum_{i=1}^{n} \vec{p}_1$$

This expression can be rewritten as,

$$\vec{p} = \sum_{i=1}^{n} m_i \vec{v}_1 = M\vec{v}_{CM}$$

where M is the total mass of the system.

"The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass."

If we differentiate the linear momentum of the center of mass with respect to time,

we obtain,

$$\frac{d\vec{p}}{dt} = M\frac{d\vec{v}_{CM}}{dt} = M\vec{a}_{CM} = \vec{F}_{ext}$$

This expression shows that if the net external force acting on a system of particles is zero ($F_{ext} = 0$ N), the linear momentum of the system is conserved.

Angular Velocity and its Relation with Linear Velocity

Angular velocity, $\omega = \frac{d\theta}{dt}$.

It is a vector quantity.

Angular velocity (ω) is related to linear velocity (v) as:

$v = r\omega$

Where r is the radius of the circular path.

$v = \omega \times r$

Linear velocity in translational motion is analogous to angular velocity in rotational motion.

Instantaneous angular velocity,

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Average angular velocity,

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous angular velocity is a vector quantity, whose direction is normal to the rotational plane and its direction is given by the right-hand screw rule.

The magnitude of angular velocity is called the angular speed, which is also represented by ω.

Angular Acceleration

The rate of change of angular velocity is called angular acceleration

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}.$$

Suppose a particle has an angular velocity $\vec{\omega}_1$ and $\vec{\omega}_2$ at time t_1 and t_2 respectively. The average angular acceleration,

$$\vec{\alpha} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1}.$$

It's a vector quantity whose direction is along the change in direction of angular velocity and the unit is rad/s^2.

Example 1.4: A wheel is rotating with an angular velocity of 2 rad/s, it is subjected to a uniform angular acceleration of 2.0 rad/s^2.

(A) What angular velocity does the wheel acquire after 10 s?

(B) How many revolutions will it make in this time interval?

Ans. (A) Given: $\omega_0 = 2$ rad/s, $\alpha = 2$ rad/s^2, $t = 10$ sec

The wheel is in uniform angular acceleration. Hence,

$\omega = \omega_0 + \alpha t$

On substituting the values of ω_0, α and t, we have

$\omega = 2 + 2 \times 10 = 22$ rad/s

$\theta = \theta_0 \times \frac{1}{2}(2 + 22)10 = 120$ rad

(B) In one revolution, the wheel rotates through 2π radians. Therefore, the number of complete revolutions n is:

$$n = \frac{\theta}{2\pi} = \frac{120}{2\pi} = 19$$

Example 1.5: A flywheel rotates with a uniform angular acceleration. Its angular velocity increases from 20π rad/s to 40π rad/s in 10 seconds. How many rotations did it undergo in this period?

Ans. As we know, $\because \omega = \omega_0 + \alpha t$

$\therefore \quad 40\pi = 20\pi + 10\alpha$

$\Rightarrow \quad \alpha = 2\pi$ rad/s^2

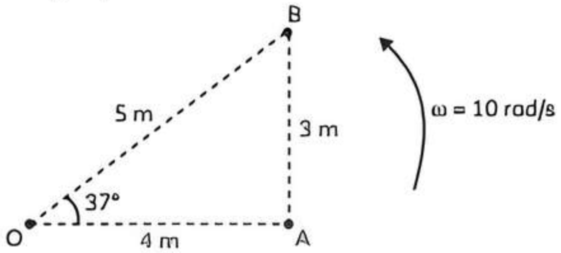
Angular displacement,

$$\theta = \frac{\omega_2^2 - \omega_1^2}{2\alpha} = \frac{(40\pi)^2 - (20\pi)^2}{2 \times 2\pi} = \frac{1200\pi^2}{4\pi} = 300\pi \text{ rad}$$

Therefore, the number of rotations undergone,

$$n = \frac{\theta}{2\pi} = \frac{300\pi}{2\pi} = 150$$

Example 1.6: A rigid lamina is rotating about an axis passing perpendicular to its plane through point O as shown in the figure. The angular velocity of point B w.r.t A is:



(a) 10 rad/s

(b) 8 rad/s

(c) 6 rad/s

(d) 0

Ans. *(a) 10 rad/s*

Explanation: In a rigid body, the angular velocity of any point with respect to any other point is constant and is equal to the angular velocity of the rigid body.

TOPIC 4

TORQUE AND ANGULAR MOMENTUM

Torque

Torque or couple is the moment of force. It is the cross product of the force with the perpendicular distance between the axis of rotation and the point of application of force with the force.

$$\text{Torque} = \vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta$$

Its S.I. unit is N.m.

The rotational analogue of force is torque or couple. couple or torque is a pair of equal and opposite forces with different lines of action. A couple produces rotation without translation.

Angular Momentum

Angular momentum is the moment of linear momentum. It is the product of the linear momentum and the perpendicular distance of the mass from the axis of rotation.

Angular momentum, $L = mvr = \vec{r} \times \vec{p} = rp\sin\theta$

Where, $\vec{r}$ = position relative to origin

$\vec{p}$ = linear momentum at position.

Its S.I. unit is kg m^2/s.

Relation between torque (τ) and angular momentum ($\vec{L}$):

$$\tau = \frac{d\vec{L}}{dt}$$

Principle of Conservation of Angular Momentum

If the external torque acting on a system of particles is zero, then the total angular momentum of the system is conserved.

If $\tau = 0$ then

$$\vec{L}_1 + \vec{L}_2 + \ldots + \vec{L}_n = \text{constant}$$

$$L = I\omega = \text{constant}, \; I_1\omega_1 = I_2\omega_2$$

Where, I is moment of inertia.

Example 1.7: Explain with a reason that if the ice melts at the polar region then moment of inertia of earth increases, angular velocity ω decreases and the day becomes longer.

Ans. If the ice in the polar region melts. The resulting water will flow towards the equator and consequently, the moment of inertia of earth will increase because more mass has accumulated at the equator which is more distant from the rotational axis as compared to the poles.

So, from the law of COAM (Conservation of angular momentum), if I increase then ω decreases. Time period (T) = $\frac{2\pi}{\omega}$ increases.

Due to increase in time period, the duration of night and day will increase.

TOPIC 5

EQUILIBRIUM OF A RIGID BODY

A rigid body is said to be in mechanical equilibrium if both its linear momentum and angular momentum are not changing with time, or equivalently, the body has neither linear acceleration nor angular acceleration.

This means;

(1) The total force, *i.e.*, the vector sum of the forces, on the rigid body is zero.

$$F_1 + F_2 + \ldots + F_n = \sum_{i=1}^{n} F_i = 0 \qquad \ldots (i)$$

If the total force on the body is zero, then the total linear momentum of the body does not change with time. Eq. (i) gives the condition for the translational equilibrium of the body.

(2) The total torque, *i.e.*, the vector sum of the torques on the rigid body is zero.

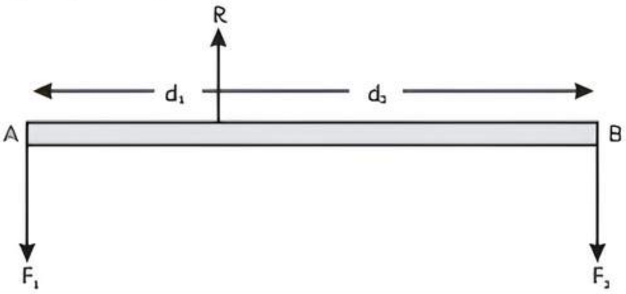
$$\tau_1 + \tau_2 + ... + \tau_n = \sum_{t=1}^{n} \tau_t = 0 \quad ... (ii)$$

If the total torque on the rigid body is zero, the total angular momentum of the body does not change with time. Eq. (ii) gives the condition for the rotational equilibrium of the body.

(3) The sum of the components of the torques along any axis perpendicular to the plane of the forces must be zero.

Principle of Moments

Consider an ideal lever, as given in the figure below. An ideal lever is nothing but a light rod (ideally negligible mass) that is pivoted at a point along its length. This point is called a fulcrum. A see-saw for children playing in the parks is an example of a lever system. The figure below shows two forces F_1 and F_2 acting on the lever. The pivot point of the lever is at a distance of d_1 and d_2 from the forces F_1 and F_2 respectively.



Assuming that the lever system is in equilibrium. Let R denote the reaction on the body, suppose at the pivot point; R is directed opposite to the forces F_1 and F_2. Since, the body is in both rotational and translation equilibrium,

$$R - F_1 - F_2 = 0 \quad ... (i)$$

$$d_1F_1 - d_2F_2 = 0 \quad ... (ii)$$

Anti-clockwise moments are taken as negative.

Equation (ii) can be re-written as,

$$d_1F_1 = d_2F_2$$

This equation is called the principle of moment's equation for the above system. The ratio $\frac{F_1}{A_1}$ and $\frac{F_2}{A_2}$ are called the mechanical advantage (M.A).

Mechanical Advantage (M.A).

$$\text{M.A.} = \frac{F_1}{F_2} = \frac{d_2}{d_1}$$

Center of Gravity

The centre of gravity of an extended body is that point where the total gravitational torque on the body is zero. In uniform gravity or gravity-free space, the centre of gravity of the body coincides with the centre of mass.

Example 1.8: A rigid body is rotating about its axis. To stop the rotation, we have to apply:

(a) pressure **(b) force**

(c) momentum **(d) torque**

Ans. *(d) torque*

Explanation: As we know that $\vec{\tau} = \frac{d\vec{L}}{dt}$

Example 1.9: If a ladder is not in the balance against a smooth vertical wall, then it can be made in balance by:

(a) decreasing the length of the ladder

(b) increasing the length of the ladder

(c) increasing the angle of inclination

(d) decreasing the angle of inclination

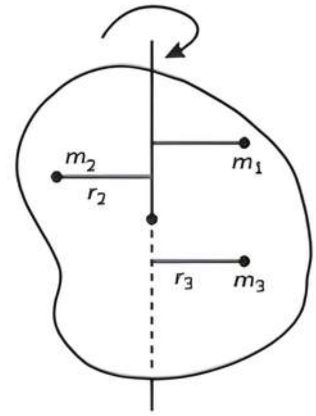
Ans. *(c) increasing the angle of inclination*

Explanation: The ladder will slip because of the gravitational force, acting on it, in the downward direction. It is given by $Mg\cos\theta$, where M is the mass of the ladder and θ is the angle of inclination. Change in length will have no effect on the force due to gravity. But when the angle of inclination is increased, it will cause the value of $\cos\theta$ to decrease and hence, reduce the force due to gravity.

TOPIC 6

MOMENT OF INERTIA

It is defined as the sum of the product of the constituent masses and the square of their perpendicular distances from the axis of rotation. The inertia of rotational motion is called the moment of inertia. It is denoted by I. Moment of Inertia is the property of an object by virtue of which it opposes any change in its state of rotation about an axis. The moment of inertia of a body about a given axis is equal to the sum of the products of the masses of its constituent particles and the square of their respective distances from the axis of rotation.



Axis of rotation

Moment of Inertia of a body is:

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ... = \sum_{i=1}^{n} m_ir_i^2$$

Its unit is kgm^2 and its dimensional formula is $[ML^2]$.

The Moment of Inertia of a body depends upon:

(1) Position of the axis of rotation
(2) Orientation of the axis of rotation
(3) Shape and size of the body
(4) Distribution of the mass of the body about the axis of rotation

The physical significance of the moment of inertia is the same in rotational motion as the mass in linear motion.

Radius of Gyration

The moment of inertia of a body about an axis is sometimes represented using the radius of gyration. It's the imaginary distance from the centroid at which the area of cross-section is imagined to be focused at a point in order to obtain the same moment of inertia. It is denoted by *k*.

The formula for the moment of inertia in terms of the radius of gyration is given as follows:

$$I = mk^2 \quad ...(i)$$

Where I, is the moment of inertia and *m* is the mass of the body.

Accordingly, the radius of gyration is given as follows:

$$K = \sqrt{\frac{I}{m}}$$

The unit of the radius of gyration is mm.

By knowing the radius of gyration, one can find the moment of inertia of any complex body with the given equation without any hassle.

Consider a body having *n* number of particles, each having a mass of *m*. Let the perpendicular distance from the axis of rotation be given by $r_1, r_2, r_3, ..., r_n$. The moment of inertia in terms of the radius of gyration is given by equation (i). Substituting the values in the equation, we get the moment of inertia of the body as follows:

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ... m_nr_n^2$$

If all the particles have the same mass, then the equation becomes:

$$I = m\frac{(r_1^2 + r_2^9 + r_3^9 + ... r_n^2)}{n}$$

From equation,

$$mk^2 = m\left(\frac{r_1^2 + r_2^9 + r_3^9 + ... + r_n^2}{n}\right)$$

or, $$k = \sqrt{\frac{r_1^2 + r_2^9 + r_3^9 + ... + r_n^2}{n}}$$

Thus, the radius of gyration can also be defined as the root-mean-square distance of various particles of the body from the axis of rotation.

Example 1.10: Four particles each of mass *m* are placed at the corners of a square of side length *l*. The radius of gyration of the system, passing through centre is:

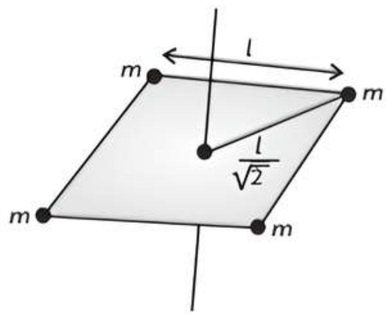
(a) $\frac{l}{\sqrt{2}}$ (b) $\frac{l}{2}$

(c) l (d) $\frac{l}{\sqrt{2}}$

Ans. (a) $\frac{l}{\sqrt{2}}$

Explanation:

$$I = m \times \left(\frac{l}{\sqrt{2}}\right)^2 \times 4$$



$$I = mk^2$$

Also, (Where *m* = 5 m)

So, $$m \times \frac{l^2}{2} = 4m \times k^2$$

$$k = \frac{l}{\sqrt{2}}$$

TOPIC 7

KINEMATICS, DYNAMICS AND ANGULAR MOMENTUM OF ROTATIONAL MOTION ABOUT A FIXED AXIS

Kinematics of Rotational Motion About a Fixed Axis

Equations of rotational motion describe the rotational motion of a body.

(1) Angular Velocity attained by a rotating body after time t

$$\omega = \frac{d\theta}{dt}$$

Average Angular acceleration of a rotating body is given by, $\vec{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$

Let at $t_1 = 0$, $\omega_1 = \omega_0$ (initial angular velocity), at $t_2 = t$, $\omega_2 = \omega$

Thus, the equation can be written as:

$$\vec{\alpha} = \frac{\omega - \omega_0}{t - 0} = \frac{\omega - \omega_0}{t}$$

$$\omega = \omega_0 + \vec{\alpha} t \quad \text{...(i)}$$

Which is angular velocity attained by a rotating body after time t.

(2) Angular velocity of a rotating body with constant acceleration after time t

$$\vec{\omega} = \frac{\theta}{t} \text{ or } \theta = \vec{\omega} t$$

If ω_0 and ω be the initial and final angular velocities of the body, respectively, then

$$\vec{\omega} = \frac{\omega_0 + \omega}{2}$$

$$\theta = \left(\frac{\omega_0 + \omega}{2}\right)t \quad \text{...(ii)}$$

We know that

$$\omega = \omega_0 + \vec{\alpha} t$$

From Eqn. (ii)

$$\theta = \frac{(\omega_0 + \omega_0 + \vec{\alpha t})}{2}$$

$$\theta = \omega_0 t + \frac{1}{2}\vec{\alpha} t^2 \quad \text{...(iii)}$$

Which is the angular displacement of the rotating body after time t.

Angular velocity of a rigid body about its axis of rotation is also expressed in terms of cross-product as

$$\vec{v} = \vec{r} \times \vec{\omega}$$

Where, $\vec{v}$ = linear velocity, $\vec{r}$ = radius vector of the particle from the axis of rotation

$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 - \omega_0^2 = 2\alpha\theta \quad \text{...(iv)}$$

Here, ω and ω_0 are final and initial angular velocities. θ = distance, α = angular acceleration

Dynamics of Rotational Motion About a Fixed Axis

In rotational motion, angular velocity, $\omega = \frac{d\theta}{dt}$

Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

Work done by torque in rotational motion $W = \tau\theta$

$$\text{K.E. of rotation} = \frac{1}{2}I\omega^2$$

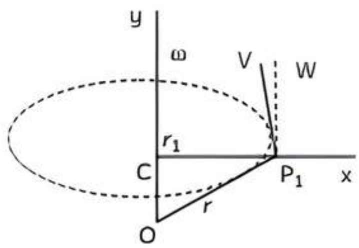
$$\text{Power, } P = \tau\omega$$

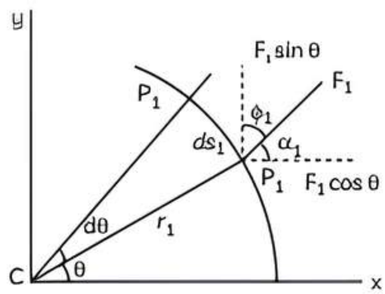
Relation between torque (τ) and angular acceleration **(α)**, $\vec{\tau} = I\vec{\alpha}$

Relation between angular momentum (L) and angular velocity **(ω)** $\vec{L} = I\vec{\omega}$

Work Done by a Torque

Consider a rigid body rotating about a fixed axis. Let F_1 be the force acting on a particle of the body at a point P_1 with its line of action in a plane perpendicular to its axis.





The particle P_1 describes a circular path of radius r_1 with centre C on its axis $CP_1 = r_1$.

At time Δt_1 the point moves to the position P_1' and displacement of the particle ds_1.

Work done by the force acting on the particle is given by,

$$dW_1 = F_1 . ds_1 = F_1\, ds_1 \cos\phi_1$$
$$= F_1 (r_1\, d\theta) \sin\alpha_1$$

Where,

ϕ_1 is the angle between F_1 and tangent P_1 and α_1 is the angle between F_1 and the radius vector OP_1.

Torque due to force F_1 about the origin is given by,

$$\tau_1 = OP_1 \times F_1$$

Effective torque due to force F_1 is given by,

$$\tau_1 = CP \times F_1$$
$$dW_1 = \tau_1\, d\theta$$

If there is more than one force acting on the body, then work done is given by,

$$dW = (\tau_1 + \tau_2 + \tau_3 + \_)\, d\theta$$
$$= \tau d\theta \qquad [\therefore \tau_1 + \tau_2 + \tau_3 + \_ = \tau]$$

Dividing both sides by dt, we get,

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

Instantaneous power, $P = \tau\omega$

Angular Momentum in Case of Rotation About a Fixed Axis

Consider a rigid body of mass M rotating with an angular velocity $\vec{\omega}$ along z-axis. The rigid body is made up of large number of elements. Consider one such element of mass m_i whose position vector is $\vec{r_i}$ and their linear momentum is $\vec{p_i}$. The angular momentum of this element about the axis of rotation is given by,

$$\vec{L_i} = \vec{r_i} \times \vec{p_i} = (r_i p_i \sin 90°)\hat{k}$$

(Angle between) $\vec{r_i}$ and $\vec{p_i}$ is 90°.

$$\vec{L_i} = (r_i p_i)\hat{k} = (r_i m_i p_i)\hat{k}$$

Since, $\quad v = r\omega,$

So, $\quad v_i = r_i\omega$

$$\vec{L_i} = (m_i r_i^2 \omega)\hat{k}$$

Therefore, the total angular momentum of the rigid body is given by,

$$\vec{L} = (\omega\hat{k})\sum m_i r_i^2$$

or, $$\vec{L} = (\omega\hat{k})I = I\omega\hat{k} = I\vec{\omega}$$

Where, I is $\sum m_i r_i^2$, a moment of inertia of a rigid body about the axis of rotation.

Which is the angular momentum of a rigid body about a fixed axis.

Conservation of Angular Momentum

The conservation of angular momentum refers to the tendency of a system to preserve its rotational momentum in the absence of external torque. For a circular orbit, the formula for angular momentum is

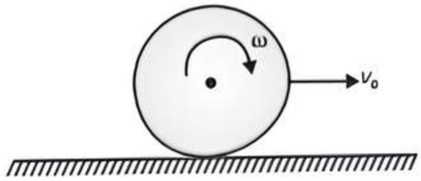
$$\frac{\text{Mass}}{\text{Times}} \times \frac{\text{Velocity}}{\text{Times}} \times \text{Radius of the circle:}$$

(Angular momentum) = $m \times v \times r$

(Angular Momentum) = (Moment of Inertia) × (Angular velocity).

Example 1.11: A solid sphere rolls without slipping on a rough surface and the centre of mass has a constant speed v_0. If the mass of the sphere is m and its radius R, then find the angular momentum of the sphere about the point of contact.

Ans.



$$\therefore \quad \vec{L}_P = \vec{L}_{cm} + \vec{r} \times \vec{p}_{cm} = I_{cm}\vec{\omega} + \vec{R} \times m\vec{v}_{cm}$$

Here $\quad v_{cm} = v_0$

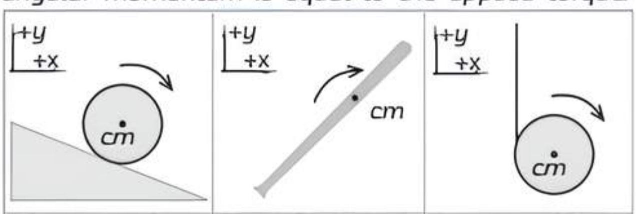
Since, the sphere is in a pure rolling motion,

Hence $\quad \omega = \frac{v_0}{R}$

$$\Rightarrow \quad \vec{L}_p = \left(\frac{2}{5} mR^2 \frac{v_0}{R}\right)(-\hat{k}) + mv_0 R(-\hat{k})$$
$$= \frac{7}{5} mv_0 R(-\hat{k})$$

Example 1.12: **Case Based:**

The total angular momentum of a particular object or system isolated from external forces is a constant, which is known as the law of conservation of angular momentum. Unless impacted by an external torque, a stiff spinning item, for example, continues to spin at a steady pace and with a set orientation. In fact, the rate of change of the angular momentum is equal to the applied torque.



A ball rolling down a ramp | A baseball bat in the air | A yo-yo up and down

(A) When a mass is rotating in a plane about a fixed point, its angular momentum is directed along:

(a) the radius

(b) the tangent to orbit

(c) line at an angle of 45° to the plane or rotation

(d) the axis of rotation

(B) A fan of a moment of inertia 0.6 kgm^2 is to run up to a working speed of 0.5 revolutions per second. Indicates the correct value of the angular momentum of the fan:

(a) $0.6\ kg \times \frac{meter^2}{sec}$ (b) $6\pi kg \times \frac{meter^2}{sec}$

(c) $3\pi kg \times \frac{meter^2}{sec}$ (d) $\frac{\pi}{6} kg \times \frac{meter^2}{sec}$

(C) A ring of mass 10 kg and diameter 0.4 meter is rotating about its geometrical axis at 1200 rotations per minute. What is its moment of inertia and angular momentum?

(D) Two bodies have their moments of inertia I and $2I$ respectively about their axis of rotation. If their kinetic energies of rotation are equal. What is the ratio of their angular momentum?

(E) Assertion (A): Value of radius of gyration of body depends upon axis of rotation.

Reason (R): Radius of gyration is root near square distance of particles of the body from the axis of rotation.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true and R is not correct explanation of A.

(c) A is true but R is false.

(d) A is false and R is also false.

[Delhi Gov. QB 2022]

Ans. (A) *(d) the axis of rotation*

Explanation: When a mass is rotating in a plane about a fixed point, its angular momentum is directed along the axis of rotation.

(B) *(a)* $0.6\ kg \times \frac{meter^2}{sec}$

Explanation: Angular momentum.

$$I\omega = 0.6 \times 0.5 \times 2\pi = 0.6\pi\ kg$$

(C) As moment of Inertia is,

$$I = mR^2 = 10(0.2)^2$$
$$= 0.4\ kg\text{-}m^2$$
$$\omega = \frac{1200 \times 2\pi}{60}\ rad/sec$$
$$\omega = 40\pi\ rad/s$$

(D) The relation between kinetic energy(K) and angular momentum(L) is:

$$L = \sqrt{2IK}$$

(Where, I = moment of inertia)

So. $$\frac{L_1}{L_2} = \sqrt{\frac{I_1}{I_2}} = \sqrt{\frac{I}{2I}} = 1 : \sqrt{2}$$

(E) *(a) Both A and R are true and R is the correct explanation of A.*

Explanation: Radius of gyration of a body about a given axis is equal to

$$k = \sqrt{\frac{r_1^2 + r_2^2 + \dots\dots\dots r_n^2}{n}}$$

It thus depends upon shape and size of the body, position and configuration of the axis of rotation and also on distribution of mass of body w.r.t. the axis of rotation.

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. When does a disc rotates with uniform angular velocity, which of the following is not true?

(a) The sense of rotation remains the same.

(b) Orientation of the axis of rotation remains the same.

(c) The speed of rotation is non-zero and remains the same.

(d) The angular acceleration is non-zero and remains the same.

[Delhi Gov. QB 2022, NCERT Exemplar]

Ans: *(d) The angular acceleration is non-zero and remains the same.*

Explanation: When the disc is rotated with constant angular velocity, the angular acceleration of the disc is zero. Because the angular acceleration $\alpha = \frac{\Delta\omega}{\Delta t}$

Here, ω is constant, so $\Delta\omega = 0$.

2. Which of the following is the mathematical representation of the law of conservation of total linear momentum?

(a) $\frac{dP}{dt} = 0$ (b) $\frac{dF}{dt} = 0$

(c) $\frac{dP}{dt} = F_{Internal}$ (d) $\frac{dF}{dt} = P$

Ans. (a) $\frac{dP}{dt} = 0$

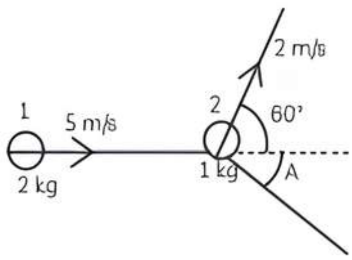
Explanation: The law of conservation of linear momentum is derived from Newton's second law. It states that total linear momentum is constant when external forces add up to zero.

According to Newton's second law: $\frac{dP}{dt} = F_{ext}$.

Here, F_{ext} must be zero for the conservation of linear momentum.

thus $\frac{dP}{dt} = 0$.

3. **Ball 1 has an initial velocity of 5 m/s in the diagram. Initially, Ball 2 is at rest. Ball 2 moves at a speed of 2 m/s in the direction depicted after the collision, while ball 1 moves at an angle of A. What is angle A's value?**



(a) $\tan^{-1}\frac{1}{\sqrt{3}}$ (b) $\sin^{-1}\frac{1}{3\sqrt{3}}$

(c) $\tan^{-1}\frac{1}{3\sqrt{3}}$ (d) $\tan^{-1}\sqrt{3}$

Ans. (c) $\tan^{-1}\frac{1}{3\sqrt{3}}$

Explanation: Linear momentum will be conserved since there is no external force. Let the speed of ball 1 be v after the collision. We will conserve momentum in the horizontal and vertical directions separately.

$2 \times 5 = 1 \times 2\cos(60) + 2 \times v\cos(A)$ and

$1 \times 2\sin(60) = 2 \times v\sin(A)$.

$\therefore \quad 9 = 2v\cos(A)$ ...(i)

$\sqrt{3} = 2v\sin(A)$ ...(ii)

On dividing the 2 equations, we get:

$$\tan(A) = \frac{1}{3\sqrt{3}}$$

$$A = \tan^{-1}\frac{1}{3\sqrt{3}}$$

4. **One circular ring and one circular disc both have the same mass and radius. The ratio of their moment of inertia about the axis passing through their centers and perpendicular to their planes will be:**

(a) 1 : 1 (b) 2 : 1

(c) 1 : 2 (d) 4 : 1

[Delhi Gov. QB 2022]

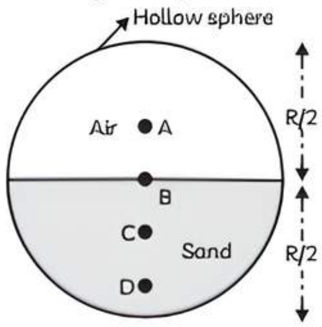
Ans. (b) 2 : 1

Explanation: Given, One circular ring and one circular disk both having the same mass and radius; Let m be the mass and n be the radius. Now, we know the moment of inertia of a circular ring about the axes passing through their center and perpendicular to its plane is $I_1 = mn^2$ moment of inertia of a circular disk about the axes passing through its center and perpendicular to its plane is $I_2 = \frac{1}{2}mn^2$

So, $$\frac{I_1}{I_2} = \frac{mn^2}{\frac{1}{2}mn^2} = \frac{2}{1}$$

$\Rightarrow \quad I_1 : I_2 = 2 : 1$

5. **Which of the following points is likely the position of the centre of mass of the system shown in the given figure?**



(a) A (b) B

(c) C (d) D [NCERT Exemplar]

Ans. (c) C

Explanation: As the air and sand are half the volume of the sphere; the volume of sand is equal to the volume of air.

The $\rho_{air} << \rho_{sand}$, $M_{sand} >> M_{air}$ inside the sphere.

As the mass of sand is larger than air, so the centre of mass will shift towards the sand from the centre of sphere B i.e., the centre mass of the system is at C.

6. **The moment of inertia of a ring about one of its diameter is I. What will be the moment of inertia about a tangent parallel to the diameter?**

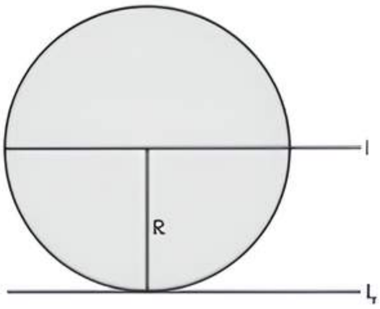
(a) 4 I (b) 2 I

(c) $\frac{3}{2}$I (d) 3 I

[Delhi Gov. QB 2022]

Ans. *(d) 3 I*

Explanation:



The moment of inertia of a ring about of its diameter is given by $I_{dia} = I = \frac{1}{2}MR^2$ where R = radius of the ring.

Here, the distance between the tangent and the diameter is R.

By the parallel axis theorem, the movement of inertia about the tangent is

$I_T = I_{dia} + MR^2 = I + 2I = 3I$ (as $MR^2 = 2I$)

7. **A thin rod of length L is suspended from one end and rotated with *n* rotations per second, the rotational kinetic energy of the rod will be:**

(a) $2mL^2\pi^2n^2$ (b) $\frac{1}{2}mL^2\pi^2n^2$

(c) $\frac{2}{3}mL^2\pi^2n^2$ (d) $\frac{1}{6}mL^2\pi^2n^2$

Ans. *(c)* $\frac{2}{3}mL^2\pi^2n^2$

Explanation: Frequency of rotation = *n* Hz

So, $\omega = 2\pi n$

And kinetic energy, $K = \frac{1}{2}I\omega^2$

$$K = \frac{1}{2} \times \frac{mL^2}{3} \times (4\pi^2 \times n^2)$$

So, $$K = \frac{2}{3}mL^2\pi^2n^2$$

⚠ Caution

➥ *Students must know that rotational Kinetic energy depends upon the axis of rotation.*

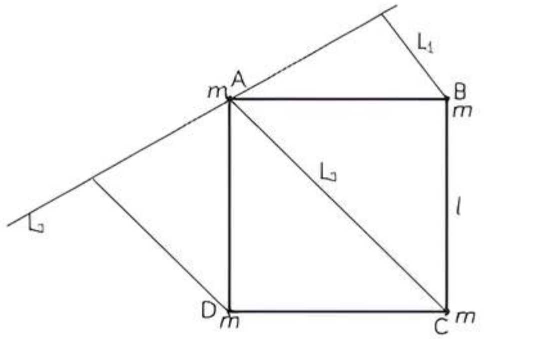
8. **Four point masses each of the value *m*, are placed at the corner of a square ABCD of side *l*. The moment of inertia of this system about an axis passing through A parallel to B D is:**

(a) $3\ ml^2$ (b) ml^2

(c) $2\ ml^2$ (d) $\sqrt{3}\ ml^2$

[Delhi Gov. QB 2022]

Ans: *(a)* $3\ ml^2$



By symmetry

$$r_1 = r_2$$

$$r_1 = \frac{1}{\sqrt{2}} = 2r$$

$$r_3 = \sqrt{2}l$$

$$MOI = \Sigma m_i r_i^2$$

$$= m\left(\frac{1}{\sqrt{2}}\right)^2 + m\left(\sqrt{2}l\right)^2 + m\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= 3\ ml^2$$

9. **A body of mass 10 kg and a radius of gyration 0.1 m is rotating about an axis. If angular speed is 10 rad/s then angular momentum will be:**

(a) $1\ kg\text{-}m^2/s$ (b) $0.1\ kg\text{-}m^2/s$

(c) $100\ kg\text{-}m^2/s$ (d) $10\ kg\text{-}m^2/s$

Ans. *(a)* $1\ kg\text{-}m^2/s$

Explanation: M = 10 kg

K = 0.1 m

ω = 10 rad/sec

Angular momentum, (L) = $I\omega$

$MK^2\ \omega = 10 \times (0.1)^2 \times 10$

$\Rightarrow$ $L = 1\ Kg\text{-}m^2/s$

Related Theory

➥ *Angular velocity of any point with respect to any other point is constant and is equal to the angular velocity of rigid body.*

10. **A wheel, having moment of inertia 2 kg-m² about its vertical axis, rotates at the rate of 60 rpm about the axis. The torque which can stop the wheel's rotation in one minute would be:**

(a) $\frac{\pi}{12}$ N–m (b) $\frac{\pi}{15}$ N–m

(c) $\frac{\pi}{18}$ N–m (d) $\frac{2\pi}{15}$ N–m

Ans. *(b)* $\frac{\pi}{15}$ N–m

Explanation: $F = 60 \text{ rpm} = \frac{60}{60} \text{ rps} = 1$

$$I = 2 \text{ kg-m}^2$$

$$\omega = \omega_0 + \alpha t$$

$$0 = 2\pi + \alpha(60)$$

$$\alpha = -\frac{2\pi}{60}$$

$$= -\frac{\pi}{30}$$

$\therefore$ (Retardation)

$$\tau = I\alpha$$

$$\tau = 2 \times \frac{\pi}{30}$$

$$= \frac{\pi}{15}$$

Related Theory

➥ *When a number of forces act on a rigid body and the body is in equilibrium, then the algebraic sum of moments in the clockwise direction is equal to the algebraic sum of moments in the clockwise direction. In other words, the algebraic sum of moments due to all of the forces is zero.*

11. **The ratio of the radii of gyration of a circular disc about a tangential axis in the plane of the disc and of a circular ring of the same radius about a tangential axis in the plane of the ring is:**

(a) 2 : 1 (b) $\sqrt{5}:\sqrt{6}$

(c) 2 : 3 (d) $1:\sqrt{2}$

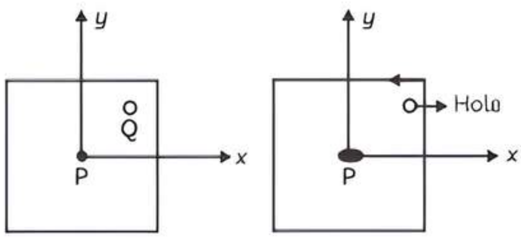
Ans. *(b)* $\sqrt{5}:\sqrt{6}$

Explanation: As we know, $K = \sqrt{\frac{I}{m}}$

$$\frac{K_1}{K_2} = \sqrt{\frac{I_1}{I_2}}$$

$$= \sqrt{\frac{\frac{5}{4}MR^2}{\frac{3}{2}MR^2}} = \sqrt{\frac{5}{6}}$$

12. **A uniform square plate has a smaller piece Q of an irregular shape removed and glued to the centre of the plate, leaving a hole behind (figure). The moment of inertia about the z-axis is then:**



(a) increased
(b) decreased
(c) remains the same
(d) changed in an unpredicted manner

[NCERT Exemplar]

Ans. *(b) decreased*

Explanation: After removing the matter from Q it is stuck at P through the axis of rotation passes, but the axis of rotation does not pass through Q. So, the gap at Q will decrease the moment of inertia as mass (removed) comes closer to the axis of rotation.

Related Theory

➥ *The radius of gyration is used to compare how various structural shapes will behave under compression along an axis. It is used to predict buckling in a compression beam or member.*

13. **Moment of Inertia of a rectangular bar magnet about an axis passing through its centre and parallel to its thickness, mass of a magnet is 1 g, length is 2 cm, breadth is 1 cm and thickness is 0.5 cm.**

(a) $\frac{10}{12}$ gcm^2 (b) $\frac{5}{48}$ gcm^2

(c) $\frac{5}{12}$ gcm^2 (d) $\frac{17}{48}$ gcm^2 [Diksha]

Ans. *(c)* $\frac{5}{12}$ *gcm^2*

Explanation: Moment of inertia along the given axis will be,

$$I = \frac{1}{12} m (l^2 + b^2)$$

$$I = \frac{1}{12} \times 1 \times (2^2 + 1^2)$$

$$I = \frac{5}{12} \text{ gcm}^2$$

14. **What is the difference in angular velocity between a wristwatch's second hand of radius 1 cm and a large clock tower's second hand of radius 5 m?**

(a) The clock tower's second hand has an angular velocity that is 500 times slower than that of the wristwatch.
(b) The clock tower's second hand has an angular velocity that is 20 times faster than that of the wristwatch.
(c) The clock tower's second hand has an angular velocity that is 500 times faster than that of the wristwatch.
(d) No difference

Ans. *(d) No difference*

Explanation: The angular velocity should not change based on the radius of the second hand. No matter what size the second hand, it will still cover one revolution every minute or the 60 s. The linear velocity will be greater and the angular momentum will also be greater for the clock tower, but its angular velocity will be the same. This can be seen by looking at the equation for angular velocity:

$$\omega = \frac{\theta}{t}.$$

15. **On account of the earth rotating about its axis:**

(a) The linear velocity of objects at the equator is greater than that of objects at the poles.

(b) The angular velocity of objects at the equator is more than that of objects at the poles.

(c) The linear velocity of objects at all places on the earth is equal, but angular velocity is different.

(d) At all places, the angular velocity and linear velocity are uniform.

Ans. *(a) The linear velocity of objects at the equator is greater than that of objects at the poles.*

Explanation: During rotation angular velocity is constant, linear velocity will be more if the distance from the centre of the earth will be more.

16. **At rest, a radioactive particle with a mass of 10 g splits into two fragments (1 and 2) with a mass ratio of 2:3 each. What is the speed of the second particle if the first moves at a speed of 10 m/s?**

(a) 4.33 m/s **(b) – 4.33 m/s**

(c) – 6.67 m/s **(d) 6.67 m/s**

Ans. *(c) – 6.67 m/s*

Explanation: Let the mass of part 1 be $2x$, and the mass of part 2 be $3x$ since the masses are in the ratio of 2 : 3.

$$2x + 3x = 10 \text{ g}$$

$$\therefore \quad x = 2\,g$$

Therefore, masses of part 1 × 2 are 4 g × 6 g respectively.

The total momentum will be conserved because there is no external force. Let the velocity of 2nd part be 'v'.

$$\therefore \quad 0 = 4 \times 10 + 6 \times v$$

$$\therefore \quad v = -\frac{40}{6} = -6.67 \text{ m/s.}$$

17. **Mass of '*m*' is at each vertex of the following shapes. Which among these shapes does not have the centre of mass on the intersection of lines of symmetry?**

(a) Cuboid **(b) Equilateral triangle**

(c) Trapezium **(d) Star** **[Diksha]**

Ans. *(c) Trapezium*

Explanation: Plane of symmetry is a plane which divides any shape into 2 equal super impossible parts (much like mirror images, symmetrical), a cuboid should have 3 planes of symmetry. An equilateral triangle has three lines of symmetry. A regular trapezium or also known as a proper trapezium is a trapezium having one pair of opposite sides parallel, but the other two sides need not be equal to each other, a proper trapezium does not give any line of symmetry. The star has 5 lines of symmetry, five lines on which it can be folded so that both sides match perfectly. A common misconception found even in many glossaries and texts: Not all lines that divide a figure into two congruent parts are lines of symmetry.

18. **An inclined plane makes an angle of 30° with horizontal. A solid sphere rolling down this inclined plane has a linear acceleration of:**

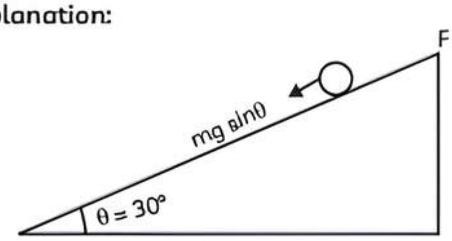
(a) $\frac{5g}{14}$ **(b)** $\frac{2g}{3}$

(c) $\frac{2}{3}$ **(d)** $\frac{5g}{7}$

[Delhi Gov. QB 2022]

Ans. *(a)* $\frac{5g}{14}$

Explanation:



$$a = \frac{mg\sin\theta - F}{m}$$

$$\alpha = \frac{FR}{I} = \frac{5FR}{2MR^2}$$

$$= \frac{5F}{2MR}$$

For no slipping, $R\alpha = a$

$$\frac{5F}{2M} = g\sin\theta - \frac{F}{M}$$

$$\frac{F}{M} = \frac{2}{7}g\sin\theta$$

$$a = g\sin\theta . \frac{F}{M} = \frac{5}{7}g\sin\theta$$

$$= \frac{5}{7}g\sin 30$$

$$= \frac{5g}{14}$$

19. A stone attached to one end of the string is revolved around a stick so that the string winds upon the stick and gets shortened. What is conserved?

(a) Angular momentum

(b) Linear momentum

(c) Kinetic energy

(d) None of the above

Ans. *(a) Angular momentum*

Explanation: Since the torque of string on a stone is zero, the angular momentum will be conserved.

Assertion-Reason Questions

Two statements are given one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes (a), (b), (c) and (d) as given below:

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true and R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false and R is also false.

20. Assertion (A): The length of the day would increase if the earth shrink.

Reason (R): It would take longer for smaller items to complete one rotation around their axis.

Ans. *(d) A is false and R is also false.*

Explanation: If the earth were to shrink, the length of the day would decrease. This is in according to the principle of conservation of angular momentum;

i.e., $I\omega$ = constant

i.e., $(MK^2)\left(\frac{2\pi}{T}\right)$ = constant

$T \propto K^2$

21. Assertion (A): Torque is due to the transverse component of force only. The radical component has no role to play.

Reason (R): This is because the transverse component is not perpendicular to the radial component.

[Delhi Gov. QB 2022]

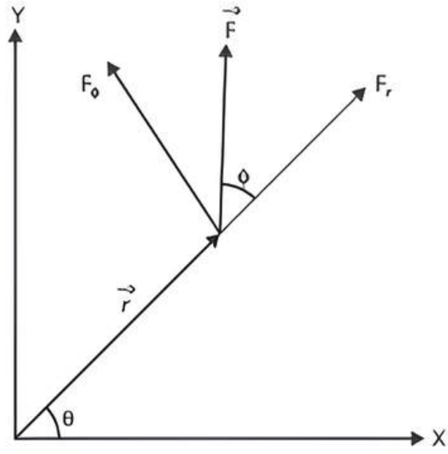
Ans. *(c) A is true but R is false.*

Explanation: As we know that,

$$\tau = \vec{r} \times \vec{F}$$

$$\tau = rF\sin\phi \quad \text{...(i)}$$

The figure given below shows the relative orientation of $\vec{r}$ and $\vec{F}$.



Resolve $\vec{F}$ into two rectangular components:

(1) $F_r = F\cos\phi$

= radial component of $\vec{F}$

(2) $F_\phi = F\sin\phi$

= transverse component of $\vec{F}$

From eq (i),

$$\tau = r(F\sin\phi)$$

$$= rF_\phi$$

i.e., Torque of a force is given by the product of the transverse component of a force and perpendicular distance from the axis of rotation. The radial component has no role play in torque.

22. Assertion (A): The condition of equilibrium for a rigid body is:

Translational motion-Σ F = 0, (*i.e.,* the sum of all external forces equal to zero).

Rotational motion-Σ τ = 0, (*i.e.,* the sum of all external torque equal to zero).

Reason(R): Under the operation of two equal and opposite forces, a rigid body must be in balance.

Ans. *(c) A is true but R is false.*

Explanation: Under the operation of two equal and opposite forces, a rigid body must be in balance.

Condition for mechanical equilibrium:

(1) The total force *i.e.,* the vector sum of the forces on the rigid body is zero.

(2) The total torque *i.e.,* the vector sum of the torques on the rigid body is zero.

$$\vec{F}_1 + \vec{F}_2 + ... \vec{F}_n = 0$$

$$\vec{\tau}_1 + \vec{\tau}_2 + ... \vec{\tau}_n = 0$$

If the forces on the rigid body are acting in the 3 dimensions, then six independent conditions are to be satisfied for the mechanical equilibrium of a rigid body.

If all the forces acting on the body are co-planar, then we need only three conditions to be satisfied for mechanical equilibrium.

A body may be in partial equilibrium *i.e.*, it may be in translational equilibrium and not in rotational equilibrium and not in translational equilibrium.

23. Assertion (A): A pipe wrench with a longer arm is required to unscrew a rusty nut.

Reason (R): The force imported to the arm is reduced by using a wrench with a longer arm.

Ans. *(c) A is true but R is false.*

Explanation: Torque is the quantity of force that can bring about an entity to rotate about the axis. Force is what makes an entity speed up in extended kinematics. Likewise, the torque is what generates angular acceleration. That is, torque can be explained as the rotational comparable of linear force. The torque or moment of force is the product of force and perpendicular distance from the pivot. Therefore, the longer the arm it will give more torque for small applying force and more torque will help to open the screw easily.

24. Assertion (A): Many great rivers flow towards the equator. The small particle that they carry increases the time of rotation of the earth about its axis.

Reason (R): The angular momentum of the earth about its rotation axis is conserved

Ans. *(a) Both A and R are true and R is the correct explanation of A.*

Explanation: In the absence of external torque, the angular momentum of the system is conserved.

$$L = I\omega = \text{constant}$$

Due to the flow of the river towards the equator, the sediments also move towards the equator which as a result increases the Moment of inertia (I) of the earth, thus decreasing angular velocity ω of the earth so that angular momentum remains constant.

As $\omega = \frac{2\pi}{t}$

Hence, when ω decreases, then the time period (T) of rotation increases.

25. Assertion (A): The center of mass of a body may lie where there is no mass.

Reason (R): The center of mass has nothing to do with the mass of the body.

[Delhi Gov. QB 2022]

Ans: *(a) Both A and R are true and R is the correct explanation of A.*

Explanation: Like Center of mass of the ring lies at its geometric center where even there is no mass of any part of the ring. So, both the Assertion and reason statements are correct and reason explains the assertion very well.

26. Assertion (A): A wheel moving down a perfectly frictionless inclined plane will undergo slipping.

Reason (R): For pure rolling, the work done against frictional force is zero.

Ans. *(b) Both A and R are true and R is not the correct explanation of A.*

Explanation: Rolling occurs only on account of friction which is a tangential force capable of providing torque when the inclined plane is perfectly smooth, it will simply slip under the effect of its weight. Once the perfect rolling begins, the force of friction becomes zero. Hence, the work done against friction is zero.

27. Assertion (A): The speed of a whirlwind in the tornado is alamingly high.

Reason (R): If no external torque acts on a body, its angular velocity remains constant.

[Delhi Gov. QB 2022]

Ans. *(c) A is true but R is false.*

Explanation: In a whirlwind in a tornado, the air from nearby regions gets concentrated in a small space thereby decreasing the value of its moment of inertia considerably. Since, $I\omega$ = constant, so due to the decrease in the moment of inertia of the air, its angular speed increases to a high value. If no external torque acts, then

$$\tau = 0 \text{ or } \frac{dL}{dt} = 0 \text{ or } L = \text{constant}$$

or $$= I\omega = \text{constant}$$

As in the rotational motion, the moment of inertia of the body can change due to the change in position of the axis of rotation, the angular speed may not remain conserved.

28. Assertion (A): Moment of inertia is always constant.

Reason (R): Angular moment is conserved, that is why the moment of inertia is constant.

Ans. (a) *Both A and R are true and R is the correct explanation of A.*

Explanation: The moment of inertia is not a fixed quantity, but depends on the orientation and position of the axis of rotation with respect to the body as a whole.

29. **Assertion (A):** The spokes near the top of a rolling bicycle wheel are more blurred than those near the bottom of the wheel.

Reason (R): The spokes near the top of the wheel are moving faster than those near the bottom of the wheel.

Ans. (a) *Both A and R are true and R is the correct explanation of A.*

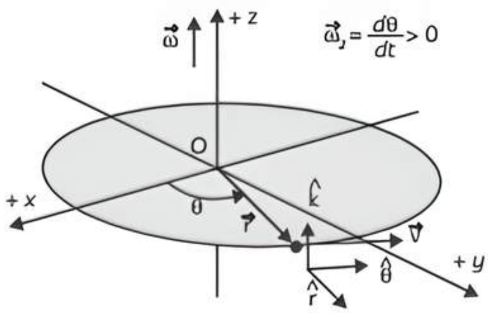
Explanation: The spokes near the top of the wheel are more blurred than those near the bottom of the wheel because they are moving faster. The wheel pushes on the ground at a certain speed to move the bike in an opposite direction but with an equal speed. The bike and thus the bottom of the wheel translates at a speed equal to but in an opposite direction to the rotational velocity of the bottom of the wheel. The pure rotation and the pure translation cancel each other, yielding a net velocity of zero at the bottom of the wheel. Because the very bottom of the wheel has a zero net velocity, the camera captures it sharply.

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

30. *Every particle of a rotating body moves in a circle. The linear velocity of the particle is related to the angular velocity. The relation between these two quantities involves a vector product for rotation about a fixed axis, the direction of the vector ω does not change with time. Its magnitude may, however, change from instant to instant. For the more general rotation, both the magnitude and the direction of ω may change from instant to instant.*



(A) What is the angular velocity of a bike wheel if it takes 3 seconds to complete one revolution?

(B) A 95 kg person is riding a Ferris wheel with a radius of 10 m. The wheel rotates at a constant angular rate of one revolution per minute. Determine the rider's linear velocity.

(C) Two men stand facing each other on two separate boats drifting on quiet water. Both are holding the end of a rope. Why do the two boats always meet at the same location, whether each guy pulls alone or both full together? Will the time taken in the two scenarios be different? Ignore friction.

Ans. (A) The definition of angular velocity is

$$\omega = \frac{\Delta\theta}{\Delta t}$$

By identifying the given information to be $\Delta\theta = 2\pi$ and $\Delta t = 3s$, we can plug this into the equation to calculate the angular velocity:

$$\omega = \frac{2\pi}{3s} = 2.09\ s^{-1}$$

(B) Convert $\frac{\text{rotations}}{\text{minute}}$ to $\frac{\text{meters}}{\text{second}}$:

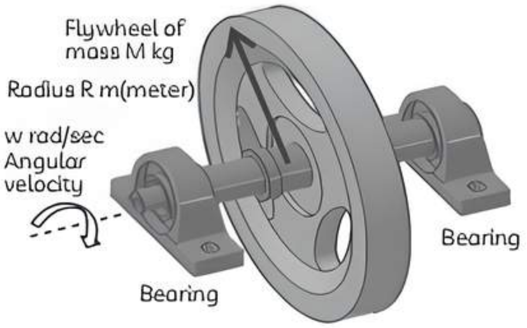
$$= 1\frac{\text{rotations}}{\text{minute}} \times \frac{1\text{minute}}{60\text{seconds}}$$

$$= \frac{2\times\pi\times 10\text{metres}}{1\text{rotation}}$$

$$= \frac{1.05\text{metres}}{\text{second}}$$

(C) The men on the two boats floating on water constitute a single system. So, the forces applied by the two men are internal. Whether each man pulls separately or both pull together, the centre of mass of the system of boats remains fixed due to the absence of any external forces. Consequently, the two boats meet at a fixed point, which is the centre of mass of the system.

31. *A heavy wheel called a flywheel is linked to the shaft of a steam engine, automobile engine, and so on. Because of its huge moment of inertia, the flywheel opposes the vehicle's sudden increase or drop in speed. It enables a gradual shift in speed and prevents jarring motion, ensuring a comfortable ride for passengers.*



(A) A flywheel is so constructed that almost the whole of its mass is concentrated at its rim, because:

(a) it increases the moment of inertia of the flywheel

(b) it decreases the moment of inertia of the flywheel

(c) it increases the speed of the flywheel

(d) it increases the power of the flywheel

(B) Assertion (A): A wheel moving down a perfectly frictionless inclined plane shall undergo slipping (not rolling).

Reason (R): For rolling torque is required, which is provided by tangential frictional force.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true and R is not correct explanation of A.

(c) A is true but R is false.

(d) A is false and R is also false.

[Delhi Gov. QB 2022]

(C) A circular disc constructed of iron and aluminium must have the greatest moment of inertia about the geometrical axis. How this is feasible?

(a) Aluminium at interior surrounded to it.

(b) Iron at the interior and aluminium surrounding to it.

(c) Using iron and aluminium layers in an alternative order.

(d) Sheet of iron is used at both the external surface and aluminium sheet's internal layer.

(D) A ferries wheel has a trip duration of 3 minutes, which means it takes three minutes to complete one full revolution. What is the angular velocity of the ferries wheel in rad/s if it only takes people around once?

(a) 18.85 (b) 0.35

(c) 0.035 (d) 2.094

(E) A CD rotates at a rate of 5 rad/s in the positive counterclockwise direction. After pressing play, the disk is speeding up at a rate of 2 rad/s^2. What is the angular velocity of the CD in rad/s after 4 seconds?

(a) 13 (b) 19.5

(c) 6 (d) 10

Ans. (A) *(a) it increases the moment of inertia of the flywheel*

Explanation: The concentration of the mass at the rim increases the moment of inertia (I) of the flywheel. Such a wheel gains or loses some kinetic energy of rotation $\frac{1}{2}I\omega^2$. It brings about a relatively smaller change in its angular speed ω. Hence, such a flywheel helps in maintaining uniform rotation.

(B) *(b) Both A and R are true and R is not correct explanation of A.*

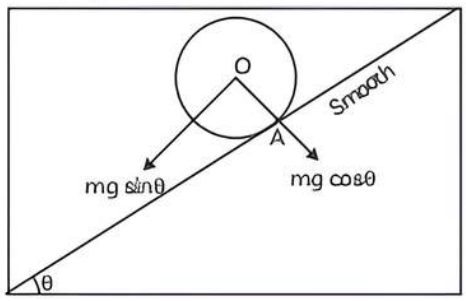
Explanation: As the surface of the inclined plane is smooth, there is no friction between it and the wheel.

The wheel rolls along the gradient as a result of *mg*sinθ.

A body's rotational motion is caused by torque. Yet, there is no torque occurring on the wheel around point O. As a result, there is no rotational motion on the wheel.

As a result, the wheel moves downward on the inclined plane surface.

The point of contact between the surface and the body is always at rest during a pure rolling action. As a result, there is no labor against the friction.



(C) *(a) Aluminium at interior surrounded to it.*

Explanation: Moment of inertia depends on the distribution of mass about the axis of rotation. Density of iron is more than that of aluminium, therefore for moment of inertia to be maximum, the iron should be far away from the axis. Thus, aluminium should be at the interior and iron surrounds it.

(D) *(c) 0.035*

Explanation: Angular velocity, in rad/s, is given by the length travelled divided by the time taken to travel the length:

$$\omega = \frac{\theta}{t}.$$

The amount of time taken to make one revolution is 3 min. One revolution is equal to 2π rad, and 3 minutes is equal to 180 seconds. Divide the radian value by the seconds, value to get the angular velocity.

$$\omega = \frac{2\times\pi}{180}$$

$$\omega = 0.035 \text{ rad/s}$$

(E) *(a) 13*

Explanation: Given initial angular velocity, angular acceleration, and time we can easily solve for final angular velocity with:

$$\omega = \omega_0 + \alpha t$$

$$\omega = 5 \text{ rad/s} + 2 \text{ rad/s} (4)$$

$$= 13 \text{ rad/s}$$

32. *A Point object is only a hypothetical concept. In actual practice, we have a bodies or object which have a definite size. An extended object or a real object is made up of large number of particles. Whereas a point object/ mass can have only translation motion, an extended object can have translational motion, rotational motion and a combination of translational and rotational motion as well.*

The motion of the system of particles or an extended object is quite complicated.

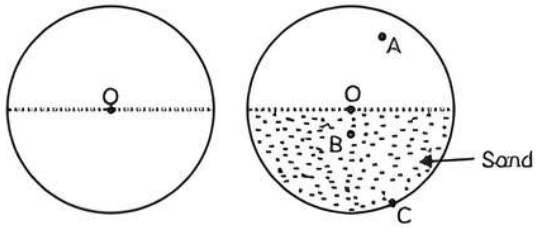
This is because every particle of the system moves in a different manner than the other particles of the system. Therefore it describe the overall motion of a body or a system of particles in a simple manner. We define the concept 'Centre of Mass'.

[Delhi Gov. QB 2022]

(A) **The center of mass of a system of particles does not depend on:**

(a) position of particles
(b) relative distance between the particles
(c) forces acting on the particles
(d) mass of the particles

(B) **The center of mass of a hollow sphere is at its center. Centre of mass of the hollow sphere when filled half with sand:**



(a) shifts to A
(b) shifts to B
(c) shifts to C
(d) remains at O (center of the sphere)

(C) **For which of the following does the center of mass lie outside the body:**

(a) a pencil (b) a dice
(c) a bangle (d) a shot put

(D) **Assertion (A):** Moment of inertia of a body is same, whatever be the axis of rotations.

Reason (R): Moment of inertia depends only a distribution of mass.

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true and R is not correct explanation of A.
(c) A is true but R is false.
(d) A is false and R is also false.

(E) **A couple is acting on a two-particle system. The resultant motion will be:**

(a) purely rotational motion
(b) purely translatory motion
(c) both (a) and (b)
(d) neither (a) nor (b)

Ans. (A) *(c) forces acting on the particles*

Explanation: The position of centre of mass is given by:

$$\vec{r}_{com} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \text{......}}{m_1 + m_2 + m_3 + \text{......}}$$

Where r_1, r_2, r_3.........

represents position of m_1, m_2, m_3. ...... respectively

From above formula it is clear that $\vec{r}_{com}$ is independent of force.

Option (c) will be correct option.

(B) *(b) shifts to B*

Explanation: The hollow sphere's center will have gravity when no sand is thrown into it. Thus, when we begin to pour sand, the bottom begins to fill. Because there is more mass concentrated at the bottom in the initial condition, the center of gravity will begin to shift from its starting position downward. The center of gravity rises when the sphere is halfway full because mass accumulates on the upper half. Once the sphere is completely filled with sand, the center of gravity will once more be at the center.

(C) *(c) a bangle*

Explanation: A bangle has a ring-like form, and its center of mass, which is outside the ring or bangle, is where the ring's center of mass is located.

(D) *(b) Both A and R are true and R is not the correct explanation of A.*

Explanation: We know that

$$I = \frac{m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^3 + \_\_ + m_n r_n^2}{m_1 + m_2 + m_3}$$

Where r_1, r_2, r_3 are distances of mass m_1, m_2, m_3 etc. from the axis. From the relation it is clear that it depends upon distribution of the masses and position of axis.

So, assertion is true.

We know that

Angular momentum = $I\omega$

Torque = $I\alpha$

If we compare these equations with equations like linear momentum = mv, force = ma, we find that I represents mass in angular motion. As mass represents inertia in linear motion. I represents inertia in angular motion.

But assertion and reason are mutually exclusive. So, (b) is the answer.

(E) *(a) purely rotational motion.*

Explanation: A pair is made up of two equal and opposing forces with parallel action axes and some lateral separation. As a result, a couple's net force (or resultant) is a null vector; as a result, there won't be any translational acceleration and just rotational motion.

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

33. A car containing explosives, goes over a ramp of 45° angle and an initial velocity of 20 m/sec. After 2 seconds, an explosion occurs. What can be said about the trajectory of the centre of mass of the car after an explosion occurs as compared to the trajectory of the car without explosion? [Diksha]

Ans. Even though each piece of the car falls apart and moves away from each other rapidly, the net acceleration acting on the centre of the mass of the car (pieces of the car) will be zero. Internal forces (explosion) do not alter the horizontal range of the centre of mass of the car.

34. A passenger is sitting in a cabin of a train that is going at a constant speed on a smooth track. What is the velocity of the compartment's centre of mass if the person starts running in it?

Ans. Velocity of the centre of mass of a system changes only when an external force acts on it. The person and the compartment form one system on which no external force is applied when the person begins to run. Therefore, there will be no change in the velocity of the centre of mass of the compartment.

35. If no external torque acts on a body, will its angular velocity be constant? [Delhi Gov. QB 2022]

Ans. A body in elliptical orbit experiences radial forces from the foci of the orbit which provides no torque. Thus, angular momentum remains constant.

$$\therefore \quad \tau = \frac{dl}{dt}$$

$$\tau = 0 \Rightarrow L = I\omega = \text{constant}$$

36. Explain with a reason why a ballet dancer varies her angular speed by outstretching her arms and legs. [Diksha]

Ans. Ballet dancers while performing folds her arms to spin faster. Thus, the act involves the use of rotational motion *i.e.,* when the dancer folds her arm while spinning the moment of inertia decreases. As $L = I\omega$, To keep the L constant the angular velocity ω increases and hence ballet dancer spins faster thereby enhancing the performance.

37. The centre of gravity of a body on the earth coincides with its centre of mass for a 'small' object whereas for an 'extended' object it may not. What is the qualitative meaning of

'small' and 'extended' in this regard?

For which of the following the two coincide? A building, a pond, a lake, a mountain? [NCERT Exemplar]

Ans. When the vertical height or Geometric centre of an object is very near to the surface of the earth, the object is called small. If it is larger than that, it is called extended object.

(1) Buildings (high), ponds, are small objects.

(2) Mountains and lakes are big objects, so their geometrical centre will be above and below the surface of the earth respectively, with appreciable distances, so-called extended objects.

38. Two spheres, one hollow and one solid rotate about their centres at the same angular speed. The mass and radius of both spheres are the same. Which has the greater rotational kinetic energy, if either?

Ans. The hollow sphere has a higher rotational kinetic energy. Its mass is located near the spherical surface, and the mass of the solid sphere is located throughout the volume. This results in a higher moment of inertia for the hollow sphere (rotational K.E. = $\frac{1}{2}I\omega^2$).

39. Why does a solid sphere have maller moment of inertia than a hollow cylinder of the same mass and radius, about an axis passing through their axes of symmetry? [NCERT Exemplar]

Ans. $I = \sum_{i=1}^{n} m_i r_i^2$

Moment of inertia is directly proportional to the square of the distance of the mass from the axis of rotation.

In a solid sphere whole mass is distributed from the centre to the radius of sphere r. But in a hollow sphere whole mass is concentrated near the periphery or surface of the sphere so the average value of r becomes larger in the hollow sphere as compared to a solid sphere.

40. How will you distinguish between a hard boiled egg and a raw egg by spinning each on a tabletop?

Ans. To distinguish between a hard boiled egg and a raw egg, we spin each on a table top. The egg which spins at a slower rate shall be raw. This is because in a raw egg, liquid matter inside tries to get away from its axis of rotation. Therefore, its moment of inertia increases. As $\tau = I\alpha$ = constant.

Therefore, α decreases *i.e.*, a raw egg will spin with smaller angular acceleration. The reverse is true for a hard boiled egg which will rotate more or less like a rigid body.

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

41. Two point masses of 4 kg and 6 kg are travelling in the same straight line at 3 m/s and 2 m/s, respectively. Find the velocity of their centre of mass if both masses are travelling in the same (A) or opposite (B) direction.

Ans. Given:

(A) $m_1 = 4$ kg, $v_1 = 3$ m/s, $m_2 = 6$ kg, $v_2 = 2$ m/s

Using,

$$v_{CM} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

$$v_{CM} = \frac{4\times3+6\times2}{4+6} = \frac{24}{10} = 2.4 \text{ m/s}$$

(B) In this case, $v_1 = 3$ m/s, $v_2 = -2$ m/s

$$v_{CM} = \frac{4\times3-6\times2}{4+6} = 0 \text{ m/s}$$

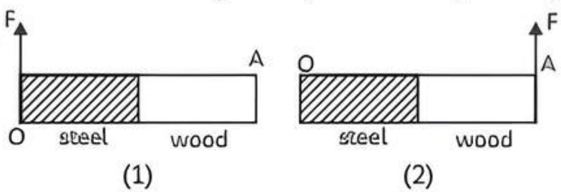
42. There is a stick half of which is wooden and half is of steel.

(A) It is pivoted at the wooden end and a force is applied at the steel end at a right angle to its length.

(B) It is pivoted at the steel end and the same force is applied at the wooden end.

In which case is the angular acceleration more and why? [Delhi Gov. QB 2022]

Ans.



In the figure (1)

Force is applied at Steel end at O.

In the second, the same force is applied at the wooden end at A.

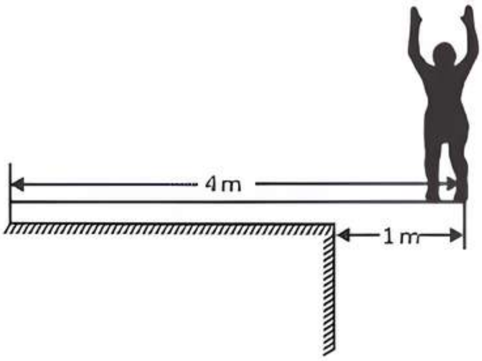
So, Torque in both the cases is same Moment of Inertia $I_1 > I_2$

(distance from heavy side with axis is greater $\tau = I\alpha$ in 1)

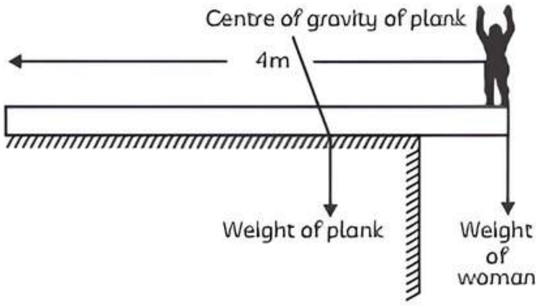
$$\tau_1 = \tau_2$$
$$I_1\alpha_1 = I_2\alpha_2$$
$$\alpha_1 < \alpha_2$$

So, the angular acceleration in the second case is greater than in the first.

43. The 40-kilogram woman stands at the end of a 4-meter-long uniform board. Estimate the mass of the plank if the maximum overhand for balance is 1 metre.



Ans.



The mass of the plank is about 40 kg. The plank tends to rotate like a seesaw about a pivot point at the edge of the building. Her weight multiplied by 1 meter produces a torque that tends to rotate the system clockwise. The counterbalancing torque is produced by the weight of the plank multiplied by the distance from the pivot point to the plank's center of gravity. The distance is also 1 meter. So, both the woman and the plank weigh the same. Their masses are equal.

44. A stationary train compartment holds 100 passengers. A violent fight breaks out among passengers due to a disagreement

(A) Will the position of the centre of mass of the compartment change?

(B) Will the position of the centre of mass of the system (compartment + 100 passengers) change?

Ans. (A) The position of the centre of mass of the compartment will change because the passengers are external bodies for the compartment.

(B) The position of centre of mass of the system will not change as no external force is acting on the system.

45. The vector sum of a system of non-collinear forces acting on a rigid body is given to be non-zero. If the vector sum of all the torques due to the system of forces about a certain point is found to be zero, does this mean that it is necessarily zero about any arbitrary point? [NCERT Exemplar]

Ans. The vector sum of all torques due to forces at a point is zero. It does not mean that the resultant of forces is zero. e.g., the torque on the see-saw of a boy and child can be equal (can be balanced). If the point of support of the see-saw changes without changing its position, the torques will not balance the see-saw. So, it is not necessary that, if the sum of all torques due to different forces at a point is zero, it will or, may not be zero for other arbitrary points.

$$G_i \sum_{i=1}^{n} F_i \neq 0$$

τ About a point P(let).

$$\tau = \tau_1 + \tau_2 + \ldots \tau_n = \sum_{i=1}^{n} r_i \times F_i = 0 \quad \text{(given)}$$

τ about any other point.

$\Sigma r_i \rightarrow$ will be different forces

$$\sum_{i=1}^{n} (\vec{r_i} - a) \times F_i = \sum_{i=1}^{n} \vec{r} \times F_i - a \sum_{i=1}^{n} F_i$$

As a and ΣF_i are not zero. So, the sum of all the torques about any arbitrary point need not be zero necessarily.

46. A wheel in uniform motion about an axis passing through its centre and perpendicular to its plane is considered to be in mechanical (translational plus rotational) equilibrium because no net external force or torque is required to sustain its motion. However, the particles that constitute the wheel do experience a centripetal acceleration directed towards the centre. How do you reconcile this fact with the wheel being in equilibrium? How would you set a half-wheel into uniform motion about an axis passing through the centre of mass of the wheel and perpendicular to its plane? Will you require external forces to sustain the motion? [NCERT Exemplar]

Ans. The wheel is a rigid elastic body. It is in uniform motion about an axis passing through its centre and perpendicular to the plane of the wheel. Each particle of the wheel which constitutes the wheel is in a circular motion about the above axis and each particle will experience a centripetal acceleration directed towards the axis of rotation due to elastic forces which are in pairs. In a half wheel, the distribution of mass of the half wheel is not symmetric about the axis of the wheel. Therefore, the direction of angular momentum and angular velocity does not coincide. Hence, the external torque is required to maintain the motion in a half wheel.

47. Show that in the absence of an external force the velocity of the C.M. of a system remains constant. [Delhi Gov. SQP 2022]

Ans. When there is no external force the acceleration of the body is zero.

The velocity of the body will be 'V'

Force = rate of change of momentum

$$\Rightarrow \quad F = \frac{d(mv)}{dt}$$

$$\Rightarrow \quad m\left(\frac{dv}{dt}\right) = F$$

The external force, $F = 0$

$$m\left(\frac{dv}{dt}\right) = 0$$

$$\frac{dv}{dt} = 0$$

m is not zero.

So, this proves that velocity will always be constant in the above situation.

48. A child stands at the centre of a turntable with his two arms outstretched. The turntable is set to rotate with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to $\frac{2}{5}$ times the initial value? Assume that the turntable rotates without friction.

Ans. Given:

Initial angular velocity, $\omega_1 = 40$ rev/min

Final angular velocity, $\omega_2 = 40$ rev/min

The moment of inertia of the boy with stretched hands $= I_1$

The moment of inertia of the boy with folded hands $= I_2$

The two moments of inertia are related as:

$$I_2 = \frac{2}{5} I_1$$

Since, no external force acts on the boy, the angular momentum L is a constant.

Hence, for the two situations, we can write:

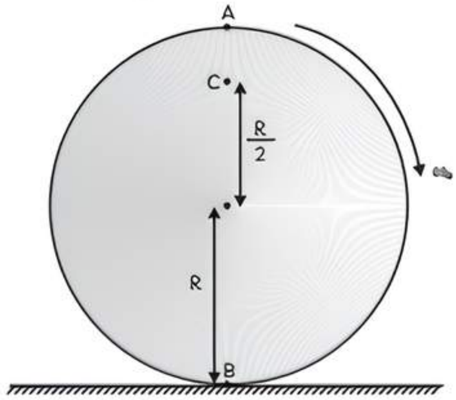
$$I_2 \omega_2 = I_1 \omega_1$$

$$\omega_2 = \frac{I_1}{I_2}$$

$$\omega_1 = \frac{I_1}{\frac{2}{5}I_1} \times 40$$

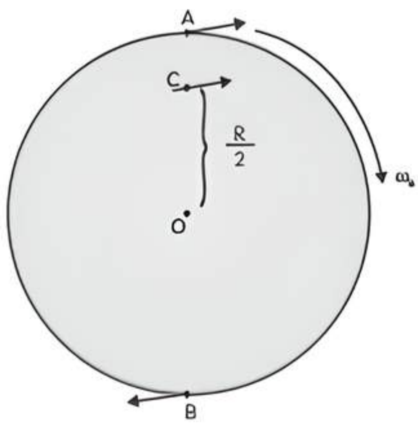
$$= 100 \text{ rev/min}$$

49. A disc rotating about its axis with angular speed ω_0 is placed lightly (without any linear push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in figure? Will the disc roll?



[Delhi Gov. QB 2022]

Ans.



Velocity of point A, $v_A = R\omega_0$ towards the right

Velocity of point B, $v_B = R\omega_0$ towards the left

Velocity of point C, $v_C = 2R\omega_0$ towards the right

Since, the disc is placed on a frictionless table, it will not roll. This is because the presence of friction is essential for the rolling of a body.

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

50. Using the expression for power and K.E. of rotational motion, derive the relation $\tau = I\alpha$. **[Delhi Gov. SQP 2022]**

Ans. We have the power of rotational motion,

$$P = \tau\omega \quad ...(i)$$

and the K.E. of rotational motion,

$$K.E. = \frac{1}{2}I\omega^2 \quad ...(ii)$$

Now, the power of rotational motion is equal to the time rate of work done during the rotational motion. Since, the work done is stored in the form of kinetic energy.

$$p = \frac{d}{dt}$$

(K.E. of rotational motion)

Using the equations (i) and (ii), we have

$$\tau\omega = \frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) = \frac{1}{2}I\frac{d}{dt}(\omega^2)$$

$$= \frac{1}{2}I(2\omega)\frac{d\omega}{dt}$$

$$= \frac{1}{2}I(2\omega)\alpha = I\omega\alpha$$

or, $\tau = I\alpha$

51. $(n - 1)$ equal point masses, each of mass m, are placed at the vertices of a regular n-polygon. The vacant vertex has a position vector $\vec{a}$ with respect to the centre of the polygon. Find the position vector of centre of mass. **[NCERT Exemplar]**

Ans. The centre of mass of a regular n-polygon lies at its geometric centre. Let $\vec{b}$ be the position vector of the centre of mass of a regular n-polygon. $(n - 1)$ equal point mass is placed at $(n - 1)$ vertices of n-polygon, then r_{cm} when mass m is placed at n^{th} vertex.

$$r_{cm} = \frac{(n-1)mb+ma}{(n-1)m+m}$$

If mass m is placed at n^{th} remaining vertex, then

$$r_{cm} = 0$$

$$\frac{(n-1)mb+ma}{(n-1)m+m} = 0$$

$$(n-1)mb+ma = 0$$

$$\vec{b} = \frac{-m\vec{a}}{(n-1)m} = \frac{-\vec{a}}{(n-1)}$$

(–) sign shows that CM lies on other side from n^{th} vertex geometrical centre of n-polygon, *i.e.*, $\vec{b}$ is opposite to the vector $\vec{a}$ from centre to n^{th} vertex.

52. A rotating table has angular velocity 'ω' and moment of inertia I_1. A person of mass 'm' stands on the centre of the rotating table. If the person moves a distance r along its radius, then what will be the final angular velocity of the rotating table?

Ans. The process takes place in the absence of external torque, so the law of conservation of angular momentum can be applied.

Initial angular momentum = Final angular momentum

$$I_1\omega_1 = (I_1 + mr^2)\omega_2$$

$$\Rightarrow \quad \omega_2 = \frac{I_1\omega}{I_1 + mr^2}$$

53. Equal torques are applied on a hollow cylinder and solid sphere, both having the same mass and radius. The cylinder rotates about its axis and the sphere rotates about its diameter. Which one will acquire greater speed and why? **[Delhi Gov. SQP 2022]**

Ans. We know $\tau = I\alpha$

or, $\alpha = \frac{\tau}{I}$

Angular acceleration produced in the cylinder is,

$$\alpha_c = \frac{\tau}{I_c}$$

Similarly, the angular acceleration produced in the sphere is,

$$\alpha_s = \frac{\tau}{I_s}$$

Thus, $\frac{\alpha_c}{\alpha_s} = \frac{I_s}{I_c}$

Now, $I_s = \frac{2}{3}MR^2$ and $I_c = MR^2$

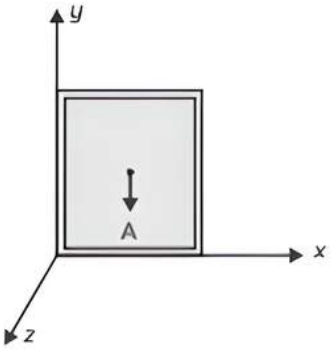
or, $\frac{\alpha_c}{\alpha_s} = \frac{2}{3}$

or, $\alpha_s = \frac{2}{3}\alpha_c$

Since, $\alpha_s < \alpha_c$

Therefore, the sphere will acquire greater speed than that of the cylinder.

54. **A door is hinged at one end and is free to rotate about a vertical axis (figure). Does its weight cause any torque about this axis? Give reason for your answer.**



[NCERT Exemplar]

Ans. We know that, $\vec{\tau} = \vec{r} \times \vec{F}$

Here, the axis of rotation of the door is along Y-axis and the door is in x-y plane and force F can be applied along ± z-axis, the torque is experienced by the door. So, a force can produce torque only along an axis in the direction normal to the force. Force due to the gravity of door is parallel to the axis of rotation. So cannot produce torque along y-axis. Gravity due to the door is along $-y$-axis.

So, it can rotate the door in the axis along ± z-axis.

Hence, the weight of the door cannot rotate the door along y-axis.

55. **At the centre, a metre stick is poised on a razor edge. When two 5 g coins are placed one on top of the other at the 12.0 cm mark, the stick is discovered to be balanced at 45.0 cm. What is the weight of a metre stick? [Diksha]**

Ans. Let W and W' be the respective weights of the metre stick and the coin.

The mass of the metre stick is concentrated at its mid-point, *i.e.*, at the 50 cm mark.

Mass of the meter stick = M'

Mass of each coin, m = 5 g

When the coins are placed 12 cm away from the end P, the centre of mass gets shifted by 5 cm from point R toward the end P. The centre of mass is located at a distance of 45 cm from point P.

The net torque will be conserved for rotational equilibrium about point R.

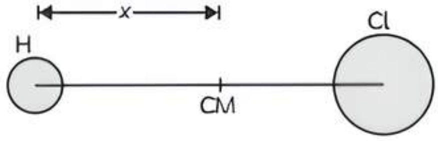
$10 \times g\,(45 - 12) - m'\,g\,(50 - 45) = 0$

$$m' = \frac{10 \times 33}{5} = 66 \text{ g}$$

Hence, the mass of the metre stick is 66 g.

56. **In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å. (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus. [Delhi Gov. SQP 2022]**

Ans. Let us choose the nucleus of the hydrogen atom as the origin for measuring distance.



Mass of hydrogen atom, m_1 = 1 unit (let)

Mass of chlorine atom, m_2 = 35.5 units (let)

Now, $x_1 = 0$ and $x_2 = 1.27$ Å

$= 1.27 \times 10^{-10}$ m

Distance of CM of HCl molecule from the origin is given by

$$x = \frac{(m_1 x_1 + m_2 x_2)}{(m_1 + m_2)}$$

$$= \frac{(1 \times 0 + 35.5 \times 1.27 \times 10^{-10})}{(1 + 35.5)}$$

$= 1.235 \times 10^{-10}$ m

$= 1.235$ Å

57. **A bullet with a mass of 10 g and a speed of 500 m/s is fired into a door and becomes embedded precisely in the centre of the door. The door is 1.0 m in width and weighs 12 kg. It is hinged at one end and rotates almost frictionless about a vertical axis. Determine the angular speed of the door shortly after the bullet has embedded itself in it.**

Ans. Mass of the bullet,

$m = 10 \text{ g} = 10 \times 10^{-3}$ kg

Velocity of the bullet,

$v = 500$ m/s

Thickness of the door,

L = 1m

Radius of the door,

$$r = \frac{1}{2}$$

Mass of the door, M = 12 kg

Angular momentum imparted by the bullet on the door, $\alpha = mvr$

$$\alpha = (10 \times 10^{-3}) \times (500) \times \frac{1}{2}$$

$= 2.5 \text{ kgm}^2\text{s}^{-1}$

Moment of inertia of the door,

$$I = \frac{1}{3} ML^2$$

$$= \frac{1}{3} \times 12 \times (1)^2 = 4 \text{ kgm}^2$$

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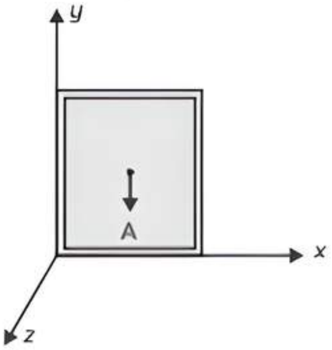

[NCERT Exemplar]

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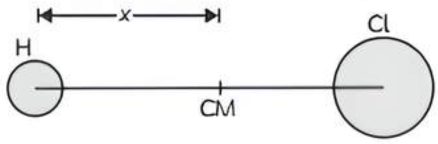
Ans. Let us choose the nucleus of the hydrogen atom as the origin for measuring distance.



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$$= 1.235 \times 10^{-10} \text{ m}$$

$$= 1.235 \text{ Å}$$

57. **A bullet with a mass of 10 g and a speed of 500 m/s is fired into a door and becomes embedded precisely in the centre of the door. The door is 1.0 m in width and weighs 12 kg. It is hinged at one end and rotates almost frictionless about a vertical axis. Determine the angular speed of the door shortly after the bullet has embedded itself in it.**

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$$= 2.5 \text{kgm}^2\text{s}^{-1}$$

Moment of inertia of the door,

$$I = \frac{1}{3}ML^2$$

$$= \frac{1}{3} \times 12 \times (1)^2 = 4 \text{ kgm}^2$$

Volume of the given parallele piped = abc

$$\vec{OC} = \vec{a}$$
$$\vec{OB} = \vec{b}$$
$$\vec{OC} = \vec{c}$$

Let $\hat{n}$ be a unit vector perpendicular to both $\vec{b}$ and $\vec{c}$. Hence, $\hat{n}$ and $\vec{a}$ have the same direction.

$$\vec{b} \times \vec{c} = bc\sin\theta\hat{n}$$
$$\vec{a}.(\vec{b} \times \vec{c}) = a.(bc\hat{n})$$
$$= abc$$

= Volume of the parallele piped

61. Two particles, each of mass *m* and speed *v*, travel in opposite directions along parallel lines separated by a distance *d*. Show that the vector angular momentum of the two particle system is the same whatever be the point about which the angular momentum is taken. [Diksha]

Ans. Vector angular momentum of the two particles system about any point A on x_1y_1.

$$\vec{L}_A = mv \times d + mv \times 0 = mv\,d$$

Similarly, vector angular momentum of the two particle system about any point B on x_2y_2.

$$\vec{L}_B = mv \times d + mv \times 0 = mv\,d$$

Let us consider any other point C on AB, where AC= x.

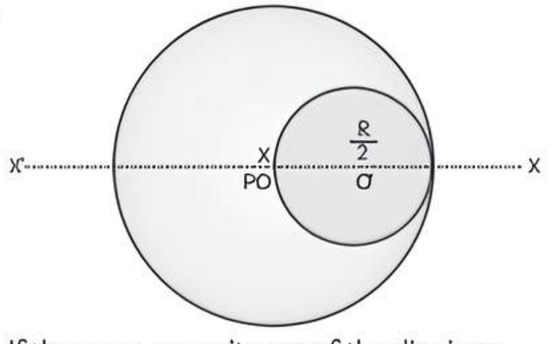
Vector angular momentum of the two particle system about C is

$$\vec{L}_C = mv(x) \times d + mv(d-x) = mv\,d$$

Clearly, $\vec{L}_A = \vec{L}_B = \vec{L}_C$

62. From a uniform disc of radius R, a circular disc of radius R/2 is cut out. The center of the hole is at R/2 from the center of the original disc. Locate the center of gravity of the resultant flat body. [Delhi Gov. QB 2022]

Ans.



If the mass per unit area of the disc is m.

Then mass of the portion removed from the disc is $M' = \pi\left(\frac{R}{2}\right)^2 \times m = \left(\frac{\pi R^2}{4}\right)m = \frac{M}{4}$

In the figure, centre of mass of M is at O and M' is at O'

But, $OO' = \frac{R}{2}$.

After mass M' is removed the remaining portion can be taken as two masses

M at O and $-M' = \frac{-M}{4}$ at O' we taking $-M'$ because we are removing this mass.

distance of centre of gravity(P) of remaining part is:

$$X = \frac{M \times 0 \quad M' \times \frac{R}{2}}{M + M'}$$

$$X = \frac{\frac{-M}{4} \times \frac{R}{2}}{M - \frac{M}{4}}$$

$$X = \frac{-R}{6}$$

Minus sign indicates that P is to the left of O.

63. Two cylindrical hollow drums of radii R and 2R, and of a common height *h*, are rotating with angular velocities ω (anti-clockwise) and ω (clockwise), respectively. Their axes, fixed, are parallel and in a horizontal plane separated by (3R + δ). They are now brought in contact (δ → 0).

(A) Show the frictional forces just after contact.

(B) Identify forces and torques external to the system just after contact.

(C) What should be the ratio of final angular velocities when friction ceases?

[NCERT Exemplar]

Ans. (A)
$$v_1 = \omega R$$
$$v_2 = 2\omega R$$

The direction of v_1 and v_2 at the point of contact C is tangentially upward. Frictional force (F) acts due to the difference in velocities of disc 1 and, F on 1 due to 2 bring F_{12} upward and F_{21} downward it will be equal and opposite by Newton's third law.

$$F_{12} = -F_{21}$$

(B) External forces acting on system are F_{12} and F_{21} which are equal and opposite so net force acting on system.

$$F_{12} = -F_{21}$$

or $$F_{12} + F_{21} = 0$$

$$|F_{12}| = |-F_{21}|$$

External torque = F × 3R (anti-clockwise)

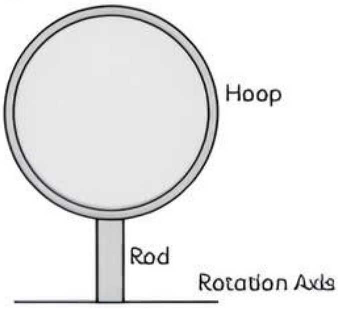
(C) Let ω_1 (anti-clockwise) and ω_2 (clockwise) be the angular velocities of drums 1 and 2 respectively.

Finally, when their velocities become equal no force of friction will act due to no slipping at this stage

$v_1 = v_2$ or $\quad \omega_1 R = 2\,\omega_2 R$

or $\quad \dfrac{\omega_1}{\omega_2} = \dfrac{2}{1}$

64. A stiff sculpture is made out of a thin hoop (mass m, radius R = 0.15 m), and a thin radial rod (mass m, length L = 2.0 R), as seen in Fig. The sculpture can pivot around a horizontal axis in the plane of the hoop, passing through its center.



Ans. A key idea here is that we can separately find the rotational inertias of the hoop and the rod and then add the results to get the sculpture's total rotational inertia

$$I_{hoop} = \frac{1}{2} mR^2.$$

The hoop has rotational inertia about its diameter. The rod has rotational inertia

$I_{cm} = \dfrac{mL^2}{12}$ about an axis through its center of mass and parallel to the sculpture's rotation axis. To find its rotational inertia I_{rot} about that rotation axis, we use the parallel-axis theorem:

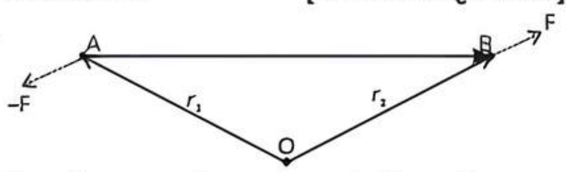
$$I_{rot} = I_{cm} + mh_{cm}^2$$
$$= \frac{mL^2}{12} + m\left(R + \frac{L}{2}\right)^2$$
$$= 4.33mR^2$$

Where, we have used the fact that, L = 2.0R and where the perpendicular distance between the rod's center of mass and the rotation axis is h = R + L/2. Thus, the rotational inertia I of the sculpture about the rotation axis is

$$I = I_{hoop} + I_{rot}$$
$$= \frac{1}{2}mR^2 + 4.33mR^2$$
$$= 4.83mR^2 \approx 4.8mR^2$$

65. Show that the moment of a couple does not depend on the point about which moment is calculated. [Delhi Gov. QB 2022]

Ans.



Consider a couple as shown in Fig. acting on a rigid body. The forces F and –F act respectively at points B and A. These points have position vectors r_1 and r_2 with respect to origin O. Let us take the moments of the forces about the origin.

The moments of the two forces making the couple

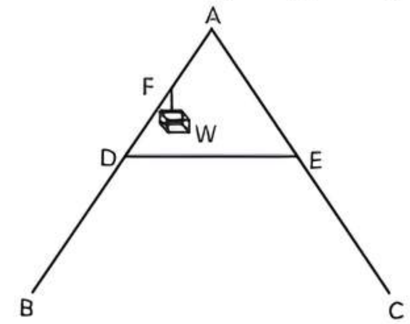
$$= r_1 \times (-F) + r_2 \times F$$
$$= r_2 \times F - r_1 \times F$$
$$= (r_2 - r_1) \times F$$

But r_1 + AB = r_2 and hence AB = $r_2 - r_1$. The moment of the couple.

therefore, is AB × F

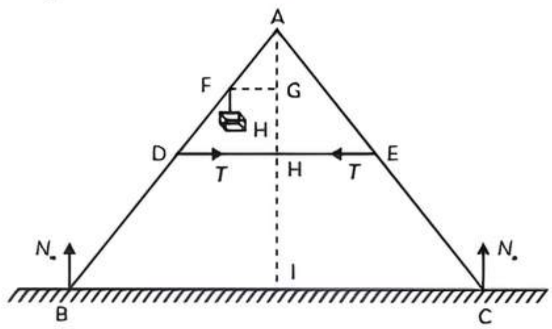
Clearly, this is independent of the origin, the point about which we took the moments of the forces.

66. As shown in the figure, the two sides of a stepladder BA and CA are 1.6 m long and hinged at A. A rope DE, 0.5 m is tied halfway up. A weight of 40 kg is suspended from point F, 1.2 m from B along the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take, g = 9.8)



(Hint: Consider the equilibrium of each side of the ladder separately.) [Diksha]

Ans. The given situation can be shown as:



N_B = Force exerted on the ladder by the floor point B

N_C = Force exerted on the ladder by floor point C

T = Tension in the rope

BA = CA = 1.6 m

DE = 0. 5 m

BF = 1.2 m

Mass of the weight, m = 40 kg

Draw a perpendicular from A on the floor BC. This intersects DE at mid-point H.

ΔABI and ΔAIC are similar

$\therefore \quad BI = IC$

Hence, I is the mid-point of BC.

$DE \parallel BC$

$BC = 2\ DE = 1\ m$

$AF = BA - BF = 0.4\ m \quad \text{... (i)}$

D is the mid-point of AB.

Hence, we can write:

$$AD = \frac{1}{2}BA = 0.8\ m \quad \text{... (ii)}$$

Using equations (i) and (ii), we get:

$FE = 0.4\ m$

Hence, F is the mid-point of AD.

FG is parallel to DH and F is the mid-point of AD. Hence, G will also be the mid-point of AH.

ΔAFG and ΔADH are similar;

$$\frac{FG}{DH} = \frac{AF}{AD}$$

$$= \frac{0.4}{0.8} = \frac{1}{2}$$

$$FG = \frac{1}{2}DH$$

$$= \frac{1}{2} \times 0.25 = 0.125$$

In ΔADH,

$$AH = \sqrt{(AD)^2 - (DH)^2}$$

$$= \sqrt{(0.8)^2 - (0.25)^2} = 0.76 m$$

For the translational equilibrium of the ladder, the upward force should be equal to the downward force.

$N_C + N_B = mg = 392 \quad \text{... (iii)}$

For the rotational equilibrium of the ladder, the net moment about A is:

$$-N_B \times BI + mg \times FG + N_c \times CI + T \times AG - T \times AG = 0$$

$$-N_B \times 0.5 + 40 \times 9.8 \times 0.125 + N_c \times (0.5) = 0$$

$$N_c - N_B = 98$$

Adding equations (iii) and (iv), we get:

$N_C = 245\ N$

$N_B = 147\ N$

For rotational equilibrium of the side AB, consider the moment about A.

NUMERICAL Type Questions

67. To maintain a rotor at a uniform angular speed of 200 rad s^{-1}, an engine needs to transmit a torque of 180 Nm. What is the power required by the engine? Assume that the engine is 100% effcient.

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Ans. Given: The angular speed of the rotor is 200 rad/s.

The torque needed to be transmitted by the engine is 180 Nm.

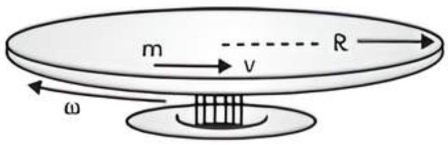
The power of the rotor is required to transmit energy to apply a torque τ to rotate a motor with angular speed ω.

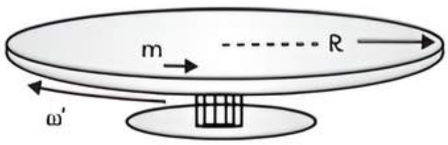
$$P = \tau\omega$$

$$= 180 \times 200\ W$$

$$= 36\ kW$$

68. A cockroach of mass *m* is moving with velocity *v* in the anticlockwise sense on the rim of a disc of radius R. the M.I. of the disc about the axis is T and it is rotating in the clockwise direction with an angular velocity 'ω'. If the cockroach stops, then calculate the angular velocity of the disc. (2m)





Ans. Angular momentum of the cockroach about the axis passing through the center of the disc:

$$L_c = mV R$$

Angular momentum of the disc $L_d = I\omega$

$$L_{total} = L_c + L_d = I\omega - mV R$$

Since, both angular momentum are opposite to each other.

When the cockroach stops moving, due to the conservation of angular momentum,

$$(I + mR^2)\omega' = I\omega - mV R$$

$$\Rightarrow \quad \omega' = \frac{I\omega - mVR}{I + mR^2}$$

69. During the launch from the board, a diver's angular speed about her centre of mass changes from zero to 6.20 rad/s in 220 ms. Her rotational inertia about the centre of mass is 12.0 kg/m^2. During the launch, what is the magnitude of:

(A) Her average angular acceleration

(B) The average external torque acting on her from the board? (2m)

Ans. (A) From the kinematic equation, $\omega = \omega_0 + \alpha t$,

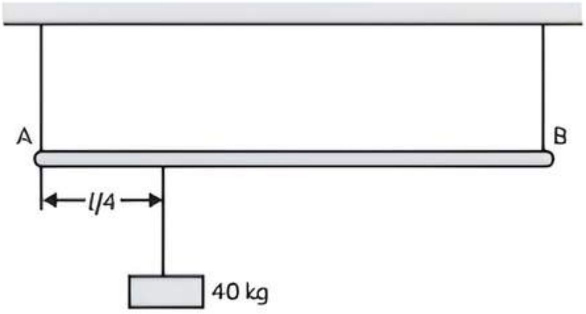
we get, $\alpha = \dfrac{\omega - \omega_0}{t}$

$$= \frac{6.20\ rad/s}{220 \times 10^3 s} = 28.2\ rad/s$$

(B) If I is the rotational inertia of the diver. Then the magnitude of the torque acting on her is,

$$\tau = I\alpha$$
$$= (12.0\ \text{kg.m}^2)(28.2\ \text{rad/s}^2)$$
$$= 3.38 \times 10^2\ \text{N-m}$$

70. A uniform rod of 20 kg is hanging in a horizontal position with the help of two threads. It also supports a 40 kg mass, as shown in the figure. Find the tensions developed in each thread. (3m)

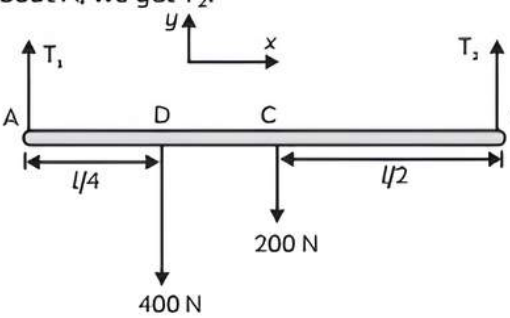


Ans. Free body diagram of the rod is shown in the figure.

Translational equilibrium requires,

$$\Sigma F_y = 0$$
$$T_1 + T_2 = 400 + 200 = 600\ \text{N} \quad \text{...(i)}$$

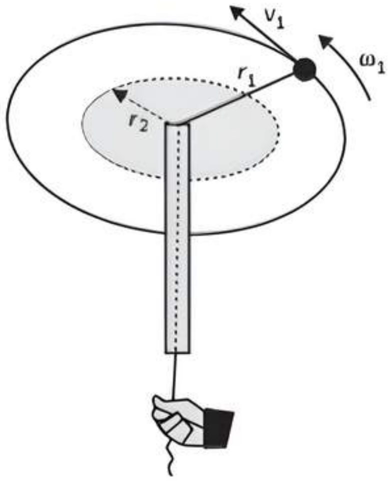
Rotational equilibrium: applying the condition about A, we get T_2.



$$\Sigma \vec{T}_A = \vec{0}$$
$$\Rightarrow \quad -400\left(\frac{l}{4}\right) - 200\left(\frac{l}{2}\right) + T_2 l = 0$$
$$T_2 = 200\ \text{N}$$

From equation (i). $\quad T_1 = 400$ N

71. A point mass is tied to one end of a cord whose other end passes through a vertical hollow tube, caught in one hand. The point mass is being rotated in a horizontal circle of radius 2 m with a speed of 4 ms^{-1}. The cord is then pulled down so that the radius of the circle reduces to 1 m. Compute the new linear and angular velocities of the point mass and also the ratio of kinetic energies in the initial and final states. (5m)



Ans. The force on the point mass due to the cord is radial, and hence the torque about the centre of rotation is zero. Therefore, the angular momentum must remain constant as the cord is shortened.

Let the mass of the particle be m. let it rotate initially in a circle of radius r_1 with linear r_2 linear velocity v_2 and angular velocity ω_2.

$\because$ initial angular momentum = final angular momentum

$$\therefore \quad I_1\omega_1 = I_2\omega_2$$
$$\Rightarrow \quad mr_1^2\frac{v_1}{r_1} = mr_2^2\frac{v_2}{r_2}$$
$$\Rightarrow \quad r_1v_1 = r_2v_2$$
$$v_2 = \frac{r_1}{v_2}v_1$$
$$= \frac{2}{1} \times 4 = 8\ \text{m/s}$$

and
$$\omega_2 = \frac{v_2}{r_2}$$
$$= \frac{8}{1} = 8\ \text{rad/s}$$

$$\frac{\text{Final K.E}}{\text{Initial K.E}} = \frac{\frac{1}{2}I_2\omega_2^2}{\frac{1}{2}I_1\omega_1^2}$$
$$= \frac{mr_2^2 \times \left[\frac{v_2}{r_2}\right]^2}{mr_1^2 \times \left[\frac{v_1}{r_1}\right]^2}$$
$$= \frac{v_2^2}{v_1^2}$$
$$= \frac{(8)^2}{(4)^2} = 4.$$